NDE of Foundations


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A Theoretical Evaluation of Guided Waves in Deep Foundations

A DISSERTATION

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ABSTRACT

A Theoretical Evaluation of Guided Waves in Deep Foundations

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A theoretical approach to non-destructive evaluation of deep foundations using guided waves has been formulated. Guided wave propagation in an infinitely long cylindrical pile embedded in soil is developed from dynamic equations of elasticity. Considering axisymmetric motion in the pile, the frequency equation for longitudinal modes is derived. The frequency equation represents a transcendental relationship between the non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$. The solution to the frequency equation is satisfied by an infinite number of modes that form branches in $\Omega - \xi a$ space. The first five branches of the longitudinal family of modes in a concrete pile embedded in soft/loose and hard/dense soils were numerically evaluated. Sensitivity analyses show that the real branches in the $\Omega - \xi a$ plane are essentially independent of the shear modulus and density of the surrounding soil, and correspond closely to the branches of longitudinal modes in a free-standing pile. However, the imaginary components of the branches in the $\Omega - \xi a$ plane are higher for soils with increased shear modulus and density.

The attenuation of the longitudinal guided wave modes is represented directly by the imaginary part of the wave number, $\xi_i$, in nepers per length. The phase and
group velocities vary with the frequency or wave number, contrary to the assumptions made in the theory of the one-dimensional approach. The power and displacement profiles of a given guided wave mode generally become more oscillatory as the order of the branch increases and as the frequency increases. For non-dimensional frequencies less than 2, the L(0,1) modes display the lowest attenuation and are easily induced in a pile, as evidenced by the popularity of the impulse response test in practice. Modes on the L(0,2) branch have the least attenuation compared to all other modes for non-dimensional frequencies above 12, and their relative simple mode shapes suggest that they are likely to be induced in practice. It is shown that locations where the group velocity is a maximum, the modes are isolated, which make it feasible for these particular modes to be propagated in practice.
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LIST OF SYMBOLS

ROMAN LETTER SYMBOLS

\( a \) = pile radius
\( A \) = cross-sectional area
\( A_{l_1} \ldots A_{l_6} \) = constants
\( b \) = plate thickness
\( B_{l_1} \ldots B_{l_6} \) = constants
\( c \) = stress wave velocity in the concrete
\( c_r \) = shear wave velocity
\( c_{l_r} \) = longitudinal wave velocity
\( C_{l_1} \ldots C_{l_6} \) = constants
\( D_{l_1} \ldots D_{l_6} \) = constants
\( E \) = Young’s Modulus
\( f \) = frequency
\( f_r \) = function of the radial coordinate
\( h_r, h_\theta, h_z \) = functions of the radial coordinate
\( H_n^{(2)} \) = Hankel function of the second kind
\( i \) = \( \sqrt{-1} \)
\( I_n \) = modified Bessel function of the first kind
\( j \) = subscript, \( j = p \) for pile and \( j = s \) for soil
\( J_n \) = ordinary Bessel function of the first kind
\( K' \) = low strain dynamic pile head stiffness
\( K_{\text{max}} \) = low strain dynamic pile head stiffness (theoretical maximum)
\( K_{\text{min}} \) = low strain dynamic pile head stiffness (theoretical minimum)
\( K_n \) = modified Bessel function of the second kind
\( L \) = length
\( L/D \) = length/diameter ratio
\( n \) = integer constant
\( N \) = mobility (mean value)
\( P \) = maximum peak resolution
\( Q \) = minimum peak resolution
\( r \) = radial coordinate
\( \hat{r} \) = radial unit vector
\( R \) = non-dimensional radius
\( \Delta t \) = time increment
\( u_j \) = displacement vector
\( u_{r_i} \) = radial displacement
\( u_{\theta_i} \) = angular displacement
\( u_z \) = axial displacement
\( \hat{u}_r, \hat{u}_{\theta_i}, \hat{u}_z \) = non-dimensional displacements
\( U \) = non-dimensional parameter
\( V \) = non-dimensional parameter
\( V/F \) = mobility (mechanical admittance) of shaft head
\( W \) = non-dimensional parameter
\( W_n \) = represents \( Y_n \) or \( K_n \)
\( x_{mn} \) = components in coefficient matrix
\( X \) = non-dimensional parameter
\( Y \) = non-dimensional wave number
\( Y_n \) = ordinary Bessel function of the second kind
\( z \) = axial coordinate
\( \hat{z} \) = axial unit vector
\( Z_n \) = represents \( J_n \) or \( I_n \)
GREEK LETTER SYMBOLS

$\alpha$ = substitution

$\beta$ = substitution

$\xi, \gamma$ = wave number

$\xi_o$ = non-dimensional wave number

$\lambda$ = wavelength

$\lambda_i$ = Lamé constant

$\mu_i$ = Lamé constant (shear modulus)

$\kappa_i$ = ratio of shear wave velocity to longitudinal wave velocity

$\eta$ = ratio of shear wave velocity in the pile to shear wave velocity in the soil

$\rho_i$ = density

$\phi_i$ = scalar potential

$\Psi_i$ = vector potential

$\psi_r, \psi_\theta, \psi_z$ = components of vector potential

$\theta$ = angular coordinate

$\hat{\theta}$ = angular unit vector

$\Theta_r, \Theta_\theta, \Theta_z$ = functions of the angular coordinate

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}$ = normal stresses

$\hat{\sigma}_{rr}, \hat{\sigma}_{\theta\theta}, \hat{\sigma}_{zz}$ = non-dimensional normal stresses

$\tau_{r\theta}, \tau_{r\theta}, \tau_{\theta\theta}$ = shear stresses

$\varepsilon_{rr}, \varepsilon_{\theta\theta}, \varepsilon_{zz}$ = normal strains

$\varepsilon_{rz}, \varepsilon_{r\theta}, \varepsilon_{\theta\theta}$ = shear strains

$\nu_j$ = Poisson’s ratio

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\( \omega \) = angular frequency

\( \Omega \) = non-dimensional frequency

\( \Delta, \) = dilatation

\( \nabla \) = gradient operator

\( \nabla^2 \) = three dimensional Laplace operator
In memory of my mother, Kathija Beegum
and my father, Haneefa

“Do ye not see that God has subjected to your (use) all things in the heavens
and on earth, and has made his bounties flow to you in exceeding measure,
(both) seen and unseen? Yet there are among men those who dispute about
God, without knowledge and without guidance, and without a Book to
enlighten them!”

The Holy Qur’an
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Non-destructive evaluation of deep foundations is normally carried out to determine the structural integrity and as-built lengths of drilled shafts or concrete piles after construction. These methods are also used to determine unknown foundation types and lengths of in-service structures. There are a variety of non-destructive tests, which include impact/sonic echo, impulse response, parallel seismic, sonic logging, ultraseismic, dynamic response methods and borehole radar. Impact/sonic echo and impulse response tests are the most widely used among these low strain tests. These surface reflection tests require access to only one side of a structure and rely on the interpretation of signals obtained from reflections resulting from wave propagation from an impact on the accessible side of the structure.

Several studies have been conducted to verify the capabilities of the surface reflection methods (Baker et al., 1993; Finno et al., 1997). These studies were performed to enable the engineering community to evaluate the reliability of the non-destructive methods. Field investigations focussing on the prediction of the size and location of anomalies in drilled shafts show that these predictions do not always give reliable results. These surface reflection methods assume one-dimensional wave propagation, necessitating the use of low frequency waves, which limits the size of the
defect that can be detected. The attenuation of stress waves that propagate along the pile limits the ability of present methods to evaluate long piles in stiff soils.

A rigorous method of analyzing elastic wave propagation in plates and rods is to solve the dynamic equations of motion using a three-dimensional elasticity approach. In the last forty years, many papers have been published dealing with guided wave propagation in plates, solid cylinders, hollow cylinders and composite cylinders (Gazis, 1959; Meeker and Meitzler, 1964; Zemanek, 1972). Guided wave analysis is complicated and the equations become intractable, especially when considering finite length cylinders and impulse loading. However, the approach in the simplest case of a traction free rod shows that different families of modes can be excited. The longitudinal modes have been used successfully to non-destructively evaluate fluid-filled pipes.

The purpose of this work is to apply the guided wave approach to model wave propagation in deep foundations for the purpose of improving non-destructive evaluation. Because there are no published solutions that modeled the problem of long, solid concrete piles embedded in soil, a detailed derivation of the general frequency equation for an infinitely long solid cylinder embedded in another material was undertaken. The decomposition of the general frequency equation for wave propagation led to three simpler cases. Longitudinal modes were considered, and the frequency equation for longitudinal modes was solved numerically to obtain the
dispersion curves for an embedded cylinder. A method of quantifying attenuation of waves due to loss of wave energy into the surrounding soil (leakage loss) was derived. The attenuation, displacements and mode shapes for selected modes were studied to identify modes that were capable of being excited in field situations and would effectively propagate in a pile.

This dissertation encompasses seven chapters. Chapter 2 presents the one-dimensional impact/sonic echo and impulse response tests that are commonly used for the non-destructive evaluation of deep foundations. It describes these methods, the interpretation of test results and their limitations. The results of two field investigations that assessed the capabilities of these tests to predict pile lengths and defect locations are summarized. A theoretical approach considering harmonic wave propagation in wave-guides, i.e., in plates and cylinders/rods, is outlined. Previous studies relating to the propagation of elastic waves in traction-free plates, rods and composite rods are described. The dispersion curves for traction-free plates and rods are shown.

Chapter 3 presents a comprehensive derivation of the frequency equation for an infinitely long cylindrical pile embedded in soil. The dynamic equations of motion are solved by a three-dimensional elasticity approach. The displacements and stresses in the pile and soil are derived in terms of scalar and vector potential functions. The general frequency equation is formulated by applying boundary conditions. The
general frequency equation is shown to separate into three simpler cases of motion. The case of axisymmetric motion generates a $4 \times 4$ determinant, representing the frequency equation for longitudinal modes.

In Chapter 4, the frequency equation for longitudinal modes is reworked and expressed explicitly in non-dimensional form. The non-dimensional variables in the frequency equation are defined in terms of the Poisson's ratios, shear moduli and densities of the pile and soil. The frequency equation is expressed as a transcendental relationship between the non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$, for a given set of geometric and physical properties of the pile and soil. The equations for axial and radial displacements and stresses in the pile and soil are derived. A normalizing factor is derived so that displacements from any mode can be compared directly. The phase and group velocities of guided wave modes are defined, and the determinations of these velocities from the dispersion curves are shown.

Chapter 5 describes the numerical evaluation of the frequency equation for longitudinal modes in a simulated cylindrical concrete pile embedded in soil. A symbolic mathematical computer code for the embedded pile was written to solve numerically the transcendental relationship between $\Omega$ and $\xi a$. A second code was developed to plot the longitudinal branches for the free (non-embedded) pile. These codes are discussed and presented in the Appendix. Typical values for elastic constants and material properties for a concrete pile and soil are considered. The
elastic parameters of the simulated soils varied in consistency from soft/loose to hard/dense. The first five branches of the longitudinal modes in an embedded pile are determined. These branches consist of complex wave numbers in the real frequency domain. The frequency equation and the embedded pile code are verified by considering the limit as the properties of the surrounding soil approach that of air, yielding the well-known solutions to the free rod. The effects of shear modulus ratio and density ratio on the dispersion curves are studied.

In Chapter 6, the wave number of guided wave modes is investigated in detail. The relative shear wave velocities of the embedded pile and soil determine whether free or leaky modes are generated and the conditions under which these modes exist for deep foundations are examined. A method of quantifying the attenuation of waves due to loss of wave energy into the surrounding soil is shown. The normalized power and displacement profiles in the pile for selected guided wave modes are presented. The displacements of selected modes with low imaginary wave number and maximum and minimum group velocities are discussed. The variation of the phase and group velocities with frequency is shown.

Chapter 7 summarizes the analytical and numerical work carried out to obtain the dispersion curves and displacements of longitudinal modes for the simulated pile embedded in soil. Conclusions are drawn based on the results presented in the preceding chapters.
2.1 INTRODUCTION

The impact/sonic echo and impulse response tests are used to non-destructively evaluate deep foundations. The test methods and the interpretation of results are briefly explained, and their limitations discussed. Two field investigations used to assess the capabilities of the tests are described. The theoretical basis for guided wave propagation in plates and rods is outlined. Previous work relating to the propagation of waves in plates, rods and composite rods are described. The dispersion curves for a free plate and a free rod are shown.

2.2 NON-DESTRUCTIVE TESTS FOR DEEP FOUNDATIONS

Non-destructive tests for deep foundations may be classified into surface reflection tests and direct transmission tests. In surface reflection tests, a stress impulse is generated on the surface of the structure and a receiver placed close to the impact measures the signals associated with reflected wave pulses. Surface reflection tests that are widely used are the impact/sonic echo and impulse response tests. In direct transmission tests, the receiver records direct signals owing to wave pulses passing through the pile from the source. Typical tests are the parallel seismic and the sonic logging tests.
Surface reflection tests are quick and inexpensive. Very little preparation of a drilled shaft or pile head is required to carry out these tests. In contrast, the direct transmission methods require either a borehole to be placed near the structure, in the case of parallel seismic tests, or tubes placed in the structure before concrete placement in the case of sonic logging tests. These methods consequently are more expensive. In this dissertation, only the surface reflection tests are discussed.

2.2.1 Impact/Sonic Echo Test

The sonic echo test is the first commercially available low strain, wave propagation test and is also known as the impact echo test. A small hammer that contains a triggering device is typically used to impact the pile head. The stress waves generated by the impact travel down the shaft and are reflected at the toe (or at a change of impedance within the shaft) as an echo and travels back to the pile head. A geophone placed on the pile head measures the velocity of the surface of the pile head. Both the direct and reflected waves are detected by the geophone. The hammer and geophone are connected to an oscilloscope or a portable PC which records the velocity-time data. A schematic set-up for the sonic echo test is shown in Figure 2-1.

Assuming one-dimensional wave propagation in the pile, the stress waves are reflected at the concrete-soil interface at the pile tip and again at the concrete-air interface at the pile head. Almost all the energy is reflected at the concrete-air
interface while at the concrete-soil interface, some energy is reflected and some energy is transmitted into the soil. These reflections are detected by the geophone until the stress waves are effectively attenuated.

Figure 2-1   A Schematic Set-up for Surface Reflection Tests
Figure 2-2 shows the results of a sonic echo test which has clear toe reflections. The distance between the impact and toe reflection signals on the time axis gives the time taken for the stress wave to travel down and back up the shaft, i.e. twice the length of the pile. The following equation gives the relationship between the pile length, \( L \), the transit time, \( \Delta t \), and the stress wave velocity in the concrete, \( c \).

\[
c = \frac{2L}{\Delta t}
\]  

(2-1)

The only measured quantity is the transit time, \( \Delta t \). If the stress wave velocity is known, then the length of the shaft can be calculated. If the length of the shaft is known, then the stress wave velocity can be calculated. The velocity of the stress wave in one dimensional stress is a function of the elastic modulus and density of the concrete. The calculated velocity can indicate the quality of the concrete.

Baker et al (1993) reported that one-dimensional stress wave velocities in the range 3800-4000 m/s would indicate good quality concrete with an ultimate compressive strength on the order of 30-35 MPa. Hearne, Stokoe and Reese (1981) correlate good quality concrete with one-dimensional stress wave velocities in the range 3500-4300 m/s. The broad stress wave velocity range shows that it cannot be directly correlated to concrete strength and is a result of variations in the mix design of concrete.
Figure 2-2  Results of a Sonic Echo Test in the Time Domain
(after Gassman, 1997)

If the length of the shaft is known, an early signal arrival means the wave pulse has encountered an impedance change other than the toe of the shaft. This may be a break in the shaft, a significant change in the cross-section, or a point at which the shaft is restrained due to a stiffer soil layer (Baker et al., 1993). If there was a break in the pile, a larger amplitude in the reflected wave is expected due to the decreased damping effect. In some cases the polarity of the reflected wave (whether positive or negative with respect to the initial impact) can indicate whether the apparent defect is due to an increase or decrease in support at that point.

Baker et al. (1993) report that there is a limiting length/diameter (L/D) ratio after which all the wave energy is dissipated and no reflected response is detected from the toe of the shaft. In this situation, the only information that can be derived is
that there are no significant defects in the upper portion of the shaft, since any such
defect closer to the head than the critical L/D ratio would reflect part of the wave
pulse. The limiting L/D ratio will vary according to the stiffness of the adjacent soils,
but a typical value in medium stiff clay is 30:1.

2.2.2 Impulse Response Test

The impulse response test is a stress wave reflection method and is also known
as the transient dynamic response test. It was modified from the vibration test
developed in France by Paquet (1968). In the vibration test, an electrodynamic
vibrator imposes a sinusoidal force of constant amplitude with frequencies increasing
from 20 Hz to 1000 Hz on the pile head. Velocity transducers (geophones) placed on
the pile record the change in velocity at the surface of the pile head. An X-Y recorder
is used to plot a signal proportional to mechanical admittance \( \left| \frac{V}{F} \right| \) against a
signal proportional to the frequency (Davis and Dunn, 1974).

The impulse response test is similar to the sonic echo test except that the force
applied to the pile head is measured. A hammer with a built-in load cell is used to
impact the pile head. The impact generates wave pulses with frequencies as high as
2000 Hz, depending on the type of hammer used. The force and velocity responses in
the time domain are recorded. A portable PC is typically used for acquiring, storing
and analyzing the data.
Unlike the sonic echo test, the data are analyzed in the frequency domain. The force and velocity signals in the time domain are converted to the frequency domain by applying a Fast Fourier Transform. The resulting velocity spectrum is divided by the force spectrum to obtain the mechanical admittance or mobility. The mobility is plotted versus frequency as shown in Figure 2-3. This plot contains information regarding the shaft length (or distance to the stress wave reflector), the dynamic pile head stiffness and shaft impedance.

2.2.2.1 Shaft Length

The resonant peaks occur at constant intervals of frequency and by measuring

Figure 2-3  Ideal Result of an Impulse Response Test in the Frequency Domain (after Davis and Dunn, 1974)
the change in frequency ($\Delta f$) between the peaks, the shaft length can be computed from

$$L = \frac{c}{2\Delta f}$$

(2-2)

where $c$ is the one-dimensional stress wave velocity in the concrete.

### 2.2.2.2 Dynamic Pile Head Stiffness

The low strain dynamic pile head stiffness ($K'$) can be calculated from the low frequency portion of the curve. It corresponds to the reciprocal of the slope of the initial part of the curve and is given as

$$K' = \frac{2\pi f_M}{\left(\frac{V}{F}\right)_M}$$

(2-3)

where $f_M$ and $\left(\frac{V}{F}\right)_M$ are the frequency and mobility at point M which lies on the initial linear portion of the mobility curve. The parameter $K'$ can be correlated to the pile stiffness obtained from axial pile load tests.
Davis and Dunn (1974) showed that the theoretical maximum stiffness \( K_{\text{max}} \) and minimum stiffness \( K_{\text{min}} \) are given by

\[
K_{\text{max}} = \frac{AE}{L} \sqrt{\frac{P}{Q}} \tanh \sqrt{\frac{P}{Q}} \tag{2-4}
\]

\[
K_{\text{min}} = \frac{AE}{L} \sqrt{\frac{Q}{P}} \tanh \sqrt{\frac{P}{Q}} \tag{2-5}
\]

where \( A \) is the cross-sectional area, \( E \) is the elastic modulus, \( P \) and \( Q \) are the maximum and minimum value of mobility from the mobility plot. The actual measured stiffness \( K' \) may be compared to the limiting stiffness values \( K_{\text{max}} \) which corresponds to a pile supported on a rigid base and \( K_{\text{min}} \) which corresponds to a pile with no base support.

### 2.2.2.3 Mobility

The shaft mobility denoted by \( N \) is given by

\[
N = \sqrt{PQ} \tag{2-6}
\]

where \( P \) and \( Q \) are the maximum and minimum mobilities in the resonant portion of
the curve. \( N \) is also defined theoretically as

\[
N = \frac{1}{\rho_c v_c A}
\]  

(2-7)

where \( \rho_c \) is the density of the concrete and \( N \) is the inverse of the shaft impedance. If the measured value of \( N \) is higher than the theoretical value, then a defect is likely to be present in the shaft. The possible causes are a smaller cross-section due to necking, low concrete density due to poor concrete, or low stress wave velocity due to poor concrete.

2.3 LIMITATIONS OF SONIC ECHO AND IMPULSE RESPONSE TESTS

Where the stress wave velocity and pile length are both unknown, the interpretation of results can become especially difficult. It can also be difficult to locate defects that are located near the toe as these defects produce reflections that could easily be interpreted as the toe itself. Necking and poor concrete both produce reductions in impedance and cause reflections but distinguishing between them is not possible. The hammer impact causes Rayleigh waves to propagate along the shaft surface which causes a noisy environment and results in problems detecting defects close to the shaft head.
To locate small defects, high frequency compression waves are required. However, the wavelengths should not be less than the shaft diameter or the assumption of one dimensional stress will not hold and reflections will occur from the shaft boundaries (Hearne et al, 1981).

The attenuation of the stress wave due to leakage into the surrounding soil is a major problem with surface reflection tests, especially in shafts with high \( L/D \) ratios and shafts embedded in stiff soils. The energy imparted to the shaft by the impact is relatively small and the damping effect of the soils around the shaft will progressively dissipate that energy as the stress wave travels down and up the shaft. In very long piles in stiff soils, the damping effect may prevent the echo from reaching the pile head or render it so weak that it is inseparable from the background noise. The signal response can be progressively amplified with time to increase information from the test but it is necessary to ensure that a real echo is being amplified and not the noise (Stain, 1987).

In impulse response tests, the resolution of the signal is defined by \( P/Q \) (the ratio of the maximum and minimum mobilities). When this ratio approaches 1.0, resonant frequencies do not appear in the mobility plot. Gassman (1997) developed a theoretical resolution chart for impulse response signals in drilled shafts for uniform soil conditions. The chart gives the ratio \( P/Q \) as a function of \( L/D \) and \( c_r/c \) for
given values of $\rho_s/\rho_c$; where $c_r$ is the shear wave velocity in soil and $c$ is the one-dimensional propagation velocity in concrete, and, $\rho_s$ and $\rho_c$ are the soil and concrete densities, respectively.

Figure 2-4 shows the resolution chart developed for the ratio $\rho_s/\rho_c = 0.8$. The chart shows that for a uniform shaft with constant $L/D$ ratio, the resolution decreases with increasing shear wave velocity of the surrounding soil, i.e. with increasing soil stiffness. Considering a shaft with a fixed diameter in uniform soil conditions where $c_r/c$ is a constant, the chart shows that the resolution decreases with increasing length.

In both tests, a hammer is used to impact the pile head. The disadvantages with this impact source are the input force is neither repeatable nor controlled. Every strike generates a different stress wave and is operator dependent. The impact also generates surface waves which can mask the arrival of compression wave reflections.

2.4 FIELD INVESTIGATION OF NDE TESTS AT TEXAS A&M

Baker et al. (1993) reported a study sponsored by the Federal Highway Administration, which assessed the use of non-destructive evaluation (NDE) methods in quality assurance programs for drilled shafts. The major objectives of the study
Figure 2-4  Ideal Resolution Chart (after Gassman, 1997)
were to evaluate existing non-destructive testing techniques for identifying defects that impact the load settlement behavior and to develop a pilot acceptance criteria for drilled shaft containing defects.

Twenty drilled shafts were constructed at four sites with different soil profiles. In California, five shafts were installed at the Cupertino site and six shafts at the San Jose site. The drilled shaft lengths varied from 25 to 62 feet. Two locations with different soil conditions were selected at Texas A&M University. Five shafts were installed in the sand site and 4 shafts in the clay site. The drilled shaft lengths ranged from 30 to 76 feet.

Planned defects were created in four shafts at the Cupertino site, three shafts at the San Jose site, four shafts at the Texas sand site and three shafts at the Texas clay site. The defects comprised of inclusion, cave-in, necking, enlargement (bulb), soft bottom, cold joint and tremie defect. Five unplanned defects were discovered at the Texas sand site and two at the clay site, and included both bulging and necking.

The participating NDT teams knew the nature and position of defects at the California sites and therefore their predictions were influenced by this prior knowledge. Therefore, the predictions agreed very well with the actual defects. The test program at the Texas site was designed for a class “A” prediction whereby the
teams were not given prior information about the locations or types of defects, or about the planned lengths of the drilled shafts. However, information regarding the locations and lengths of the reference shafts, which were not defective, were given to the teams.

Table 2-1 shows the summary of the initial predictions using sonic echo and impulse response tests at the Texas sites. The predictions from the five participants were not in close agreement with the actual defects. Thus, prior knowledge on shaft lengths was important when testing shaft integrity (Baker et al., 1993).

2.5 FIELD INVESTIGATION OF NDE TESTS AT HOUSTON

Samman and O’Neill (1996) reported a field investigation involving twenty-two drilled shafts that were constructed at the National Geotechnical Experimentation Site at the University of Houston (NGES-UH). The site consists of stiff to very stiff clay with sandy clay seams from ground level to 46 feet. This is underlain by alternating layers of dense silt and very stiff to hard sandy clay. The water table was about 6.5 feet below ground level.

The drilled shafts were constructed in two sets. Eleven shafts had nominal diameters of 18-in. and were 15 ft. long. The second set had nominal diameters of 30 in. and were 23 ft. long. All the shafts were reinforced to the full depth with 1% steel
Table 2-1 Summary of Initial Predictions from Sonic Echo and Impulse Response Tests at the Texas A&M Site (after Baker, et al., 1993)

<table>
<thead>
<tr>
<th>Participant</th>
<th>Shaft</th>
<th>actual</th>
<th>a&lt;sup&gt;c&lt;/sup&gt;</th>
<th>b&lt;sup&gt;c&lt;/sup&gt;</th>
<th>c&lt;sup&gt;c&lt;/sup&gt;</th>
<th>d&lt;sup&gt;c&lt;/sup&gt;</th>
<th>e&lt;sup&gt;c&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1/S/16.6 m</td>
<td>2-4/+20% 5.6/-70%</td>
<td>2.9/Bulb 6.4/Neck</td>
<td>3/Bulb 6.4/Neck</td>
<td>2.7/Defect 6.2/Defect</td>
<td>3.2/Bulb 6.2/Defect</td>
<td>5.5 to 13/Cave-in</td>
<td></td>
</tr>
<tr>
<td>#2/S/11.6 m</td>
<td>5.3/Tremie defect, 5.0/-45%, Soft base, Excess mudcask</td>
<td>Low-quality concrete</td>
<td>6.7/Defect</td>
<td>6.2/Defect</td>
<td>6.0/Neck</td>
<td>6.0/Neck</td>
<td></td>
</tr>
<tr>
<td>#3/S/16.5 m</td>
<td>5.6/+45% 3-8/+100% Soft base</td>
<td>2.5/Bulb 6.1/Cave-in 10.1/Bulb</td>
<td>2.2/Defect</td>
<td>???</td>
<td>3.5-10.1/Defect</td>
<td></td>
<td></td>
</tr>
<tr>
<td>#4/S/11.4 m</td>
<td>No defects</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>#5/S/16.8 m</td>
<td>8.7/-63%, 3-8/+20%, 11.5/-14.0/+20%, Soft base</td>
<td>???</td>
<td>4.1/Bulb 9.1/Defect (?)</td>
<td>1.8/Bulb 9.7/Neck</td>
<td>8.2/Neck 9.7/Bell</td>
<td>???</td>
<td></td>
</tr>
<tr>
<td>#6/C/24.1 m</td>
<td>18/-43% Soft base</td>
<td>2.4/Bulb 9.5/Neck</td>
<td>2.7/Bulb</td>
<td>10.4/Defect</td>
<td>4.3/Neck</td>
<td>???</td>
<td></td>
</tr>
<tr>
<td>#7/C/10.7 m</td>
<td>No defects</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>#8/C/12.2 m</td>
<td>6.5/-12% 12.2/+45% Soft base</td>
<td>???</td>
<td>10.7/Bulb</td>
<td>2.6/Defect</td>
<td>2.7-3.7/Bulb</td>
<td>???</td>
<td></td>
</tr>
<tr>
<td>#9/C/10.4 m</td>
<td>3.0/-50%</td>
<td>2.4/Inclusion</td>
<td>3.0/Defect 4.0/Bulb</td>
<td>2.7/Defect</td>
<td>3.7/Inclusion</td>
<td>2.3/Defect</td>
<td></td>
</tr>
</tbody>
</table>

a: Shaft No./Site (S = sand site, C = clay site)/Length
b: Depth of defect in m/percentage of cross-sectional area occupied by defect (+ = bulge; - = neck or inclusion)
c: Depth of interpreted defect/description of interpreted defect given by predictor

??? indicates that a prediction could not be made
* No defect predictions were made on these shafts, which were sound.
Nominal shaft diameters were 0.915 m.
Note: a - e denotes individual participants
area. Six 18 in.-diameter shafts were drilled under a polymer drilling slurry and the concrete placed under tremie. The remaining shafts were constructed using the dry method. The compression strength of the concrete from standard cylinder tests ranged from 21 MPa to 33 MPa at 28 days. Defects were simulated using pure gum or natural rubber cut into flat sheets about 1 in. thick. The flat sheets were circular with diameters equal to the drilled shaft. A chord was cut across the circle to give the desired percentage cross-sectional area. The modulus of elasticity of the pure gum was about 7 MPa, which is close to that of stiff clay. The impedance offered to stress waves would therefore be approximately equivalent to that due to an inclusion in the drilled shaft at the site.

The participants were informed about the site conditions, the material used to create the defects, the nominal depths and diameters of the drilled shafts, and the age of the concrete. Neither the number of defective shafts nor their shapes or locations of these defects were known to the participants. The participants were allowed to perform any type of seismic testing and analyze their data in any way. All the participants chose to report either sonic echo or impulse response tests results.

The participants were asked to evaluate each of the 22 test shafts and to provide the general method used in the detection process. If an anomaly existed, they were required to state how large the area contributing to the anomaly was relative to the
cross-sectional area of the shaft, whether it represented a reduced or increased cross-section, and the depth of the anomaly. The results are shown in Table 2-2. The actual conditions of the 22 shafts are listed together with the six reporting participants. The conditions of the shafts are indicated as "OK" or "A" to mean sound or anomalous shafts, respectively.

There was no clear difference between the accuracy of all the participants. Their accuracy of predicting sound shafts was significantly below that of predicting anomalous ones, i.e. there were numerous false positives. The size and location of the actual defects are shown in Table 2-3 along with the details of the final predictions of the participants. In general, there was no consistent correlation between either the predictions of any of the participants and the actual defects, or any noticeable consistency among the participants themselves.

2.6 WAVE PROPAGATION IN WAVEGUIDES

Guided waves are waves that propagate along a boundary or waveguide. Bulk waves (longitudinal and transverse waves), propagate in infinite media and may interact with boundaries to produce the Rayleigh surface wave, which also is a guided wave. The derivation of guided waves in free plates and cylinders can be found in Meeker and Meitzler (1964). General concepts of guided wave propagation are illustrated in the following sections.
Table 2-2 Final Predictions from Sonic Echo and Impulse Response Tests at the University of Houston Site (after Samman and O’Neill, 1996)

<table>
<thead>
<tr>
<th>Participant</th>
<th>Shaft No.</th>
<th>actual</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
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*“A” indicates an anomaly; “OK” indicates sound shaft.*
Table 2-3  Details of Final Predictions (after Samman and O’Neill, 1996)

<table>
<thead>
<tr>
<th>No</th>
<th>Actual</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13/-50%</td>
<td>1.6/-28%</td>
<td>4/+20%</td>
<td>11.5/-66%</td>
<td>2.5/+125%</td>
<td>8.8/-70%</td>
<td>6.3/-24%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12.2/+100%</td>
<td>11/+100%</td>
<td></td>
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Depth(ft)/Size (±%area) of anomalies
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Depth (ft)/Size (±% area) of anomalies
2.6.1 Wave Propagation in Plates

Rayleigh (1887) derived the equations for wave propagation along the free surface of a semi-infinite elastic half-space. These surface waves are non-dispersive and decay exponentially with distance from the free surface. The velocity of the surface wave is given by the roots of a bi-cubic expression. Love (1911) showed that surface waves with displacements perpendicular to the plane of propagation are possible in a linear elastic half-space if the half-space is covered by a layer of different material. Waves confined to the neighborhood of a surface occur not only at a free surface, but also at the interface of two half-spaces of different materials. Stoneley (1924) described waves that travel along the interface between two elastic solids. Lamb (1917) derived the equations for wave propagation in an infinite plate of finite thickness with traction free surfaces. His derivation led to the well-known Rayleigh-Lamb frequency equations. Mindlin (1960) investigated these equations in detail.

The propagation of guided waves in a plate can be explained by using the “partial wave superposition technique” by Auld (1990). Incident longitudinal and shear bulk waves interact with the boundaries of the plate and generates reflected longitudinal and shear waves. Guided waves are obtained by the superposition of these incident and reflected longitudinal and shear bulk waves. The partial wave formulation of Lamb wave modes, made up of coupled incident and reflected longitudinal and vertically polarized shear waves, is shown schematically in Figure 2-5. These guided
wave solutions may be characterized by resonant standing waves in the transverse direction of the plate, formed by the bulk wave components reconstituting themselves after consecutive reflections from the upper and lower free plate boundaries.

Considering harmonic modes propagating along the wave-guide axis in the $z$-direction, the wave number vector, $\xi$, must be equivalent to those of the component partial waves as shown in Figure 2-5, where $\alpha$ is the longitudinal wave number vector.
in the transverse (x) direction, $\beta$ is the shear wave number vector in the x-direction, $\omega$ is the angular frequency, $c_L$ is the bulk longitudinal wave velocity and $c_r$ is the bulk shear wave velocity. The transverse components of the partial wave vectors for the reflected longitudinal and shear waves, $\alpha$ and $\beta$, respectively, form the steady resonance condition in the transverse direction through the thickness of the plate (Popovics, 1994).

The symmetrical and anti-symmetrical solutions which correspond to the Rayleigh-Lamb frequency equations are obtained from Auld (1990) as follows.

**Symmetric solutions**

$$\tan\left(\frac{\beta b}{2}\right) \frac{\tan\left(\frac{\alpha b}{2}\right)}{\left(\beta^2 - \xi^2\right)^2} = -\frac{4\xi^2 \alpha \beta}{(\beta^2 - \xi^2)^2} \quad (2-8)$$

**Anti-symmetric solutions**

$$\tan\left(\frac{\beta b}{2}\right) \frac{\tan\left(\frac{\alpha b}{2}\right)}{\left(\beta^2 - \xi^2\right)^2} = -\frac{(\beta^2 - \xi^2)^2}{4\xi^2 \alpha \beta} \quad (2-9)$$
These equations may alternatively be derived from linear elastic theory with the use of potential functions. The dispersion relations for the symmetric and anti-symmetric solutions, given in terms of the angular frequency, \( \omega \), as a function of the propagating wave number, \( \xi \), can then be obtained by solving Equations (2-8) and (2-9). An infinite number of solutions for Equations (2-8) and (2-9) make up the set of dispersion curves. The dispersion curves for the symmetric and anti-symmetric modes are shown in Figure 2-6. Each point on the dispersion curve represents a guided wave mode. The displacement and stress fields associated with each guided wave mode may be calculated. If \( \xi \) is real, the mode is propagating and if \( \xi \) is imaginary, the mode is non-propagating or evanescent. Complex solutions are also possible, which would suggest a leaky mode of propagation. The frequency at which the mode changes from propagating to evanescent is called the cut-off frequency.

2.6.2 Wave Propagation in Rods/Cylinders

Pochhammer (1876) and Chree (1889) independently published the earliest formulation of longitudinal and transverse wave propagation in infinitely long circular rods with traction free surfaces, based on the dynamic equations of elasticity. Ghosh (1923) derived the dispersion equations for longitudinal wave propagation in thick and thin walled infinitely long hollow circular cylinders, with both surfaces free of traction, and with one surface traction-free and the other rigidly clamped.
Figure 2-6  Frequency Spectrum for the Rayleigh-Lamb Equations for the Free Plate, where $\Omega = \omega b/c_T$ (after Mindlin, 1960)

Bancroft (1941) published numerical solutions of the dispersion equations for infinitely long circular rods. The variation in phase velocity as a function of dimensionless propagation constant was tabulated for a wide range of values of Poisson’s ratio. Holden (1951) showed that the behavior of the various longitudinal branches in a rod is closely related to the behavior of the longitudinal branches in a plate.
Based on three-dimensional linear elastic theory, Gazis (1959) investigated the propagation of waves in hollow circular cylinders with traction free surfaces. The cases of axially symmetric waves, the limiting modes of infinite wavelength, and a special family of equi-voluminal modes were considered. Baltrukonis, Gottenberg and Schreiner (1961) studied the axial-shear vibrations of an infinitely long composite circular cylinder. The composite cylinder consists of a core and a casing which are fully bonded and made up of different materials. The authors calculated the axial shear cut-off frequencies when the casing is relatively stiff but non-rigid for both the composite hollow cylinder and composite rod.

Onoe, McNiven and Mindlin (1962) examined the relation between frequency and propagation constant for axially symmetric waves in an infinitely long circular rod and plotted the frequency spectrum for real, imaginary and complex propagation constants. McNiven, Sackman and Shah (1963) studied the dispersion of axially symmetric waves in composite elastic rods. They derived the frequency equation for a rod with a circular core of one material bonded to a circular casing of a second material of different physical properties. The lowest nine branches are presented for two cases, one with a soft core and stiff casing and another, which simulates a concrete cylinder reinforced along its axis by a steel rod.
Armenakas (1965) studied the propagation of torsional waves in composite, infinitely long, traction free circular cylindrical rods on the basis of three dimensional linear theory of elasticity. He showed that in general for composite rods, the first torsional mode may not exist uncoupled. Whittier and Jones (1967) considered axially symmetric modes in a composite cylindrical shell and developed an 8x8 determinant. He provided dispersion curves, displacement distributions and stress distributions at various frequencies for a particular example. Armenakas (1967 and 1971) treated the composite cylindrical shell and listed the elements of the 12x12 determinant. He calculated the cut-off frequencies numerically and presented the dispersion curves and displacement distributions. A detailed discussion of the asymptotic velocities is given.

Zemanek (1972) published an experimental and theoretical investigation of elastic wave propagation in a cylinder. The dispersion curves corresponding to real, imaginary and complex propagation constants for the symmetric and the first four anti-symmetric modes of propagation are plotted. The radial distributions of the axial and radial displacements and of shear and normal stresses are given for the symmetric mode.

Thurston (1978) published a tutorial review of the concepts and results needed to understand wave propagation in rods and clad (composite) rods. The review was limited to rods of circular cross-section, made from homogeneous, isotropic, linear
elastic materials. The paper describes modes that are harmonic with emphasis on the modes and ranges of parameters that are of interest for ultrasonic delay lines. A detailed procedure for obtaining the general frequency equation in rods and clad rods is included in his appendix. Eight special cases can be obtained from the general frequency equation, including a clad rod with infinite thickness cladding.

An excellent treatment of guided wave propagation in cylinders is given by Meeker and Meitzler (1964). The dynamic equations of motions are integrated by the use of potential functions. A set of potential functions that satisfy the scalar and vector wave equations are derived. The displacements and stresses in the rod are then expressed in terms of the potential functions. The application of the traction-free boundary conditions results in three homogeneous equations. This system of equations has non-trivial solutions only if its determinant vanishes. The frequency equation is obtained by setting this 3x3 determinant to zero. Since the frequency equation involves Bessel functions, there are an infinite number of solutions. These solutions emerge as the family of longitudinal, torsional and flexural branches.

The axisymmetrical modes of propagation are made up of the torsional and longitudinal family of modes. The characteristic feature of the torsional family of modes is that there are only angular displacements, \( u_\theta \), and \( u_r = u_z = 0 \). The lowest order torsional mode is non-dispersive while the higher orders are dispersive. The
The frequency equation for the longitudinal family of modes is known as the Pochhammer-Chree frequency equation and is presented in Meeker and Meitzler (1964) as

\[
-(\beta^2 - \xi^2)^2 J_0(\alpha a) J_1(\beta a) + \frac{2\alpha}{a} (\beta^2 + \xi^2) J_1(\alpha a) J_1(\beta a) - 4\xi^2 \alpha \beta J_1(\alpha a) J_0(\beta a) = 0
\]  

(2-10)

where \( \xi \) is the wave number, \( \alpha \) and \( \beta \) are substitutions used in the derivation of the frequency equation, \( a \) is the radius of the rod, and \( J_0 \) and \( J_1 \) are ordinary Bessel functions of the first kind.

The dispersion curves for longitudinal modes in a free cylinder with Poisson's ratio, \( \nu = 0.3317 \) is given in Figure 2-7. Real, imaginary and complex branches are plotted in this figure. The longitudinal mode is designated by the symbol L(0,m) where m is the integer representing the order of the branches. Thus the first or lowest branch of the symmetric mode is designated by the symbol L(0,1). The longitudinal family of modes is characterized by coupled radial displacement, \( u_r \), and axial displacement, \( u_z \).

The dispersion curves for the lowest order flexural mode is shown in Figure 2-8. The first branch of the first anti-symmetric mode (ordinary family of flexural modes) is designated by the symbol F(1,1). This family of modes has features similar
Figure 2-7  Dispersion Curves for the Symmetric Longitudinal Family of Modes for a Free Infinite Cylinder (after Zemanek, 1972)

Figure 2-8  Dispersion Curves for the Anti-Symmetric Ordinary Family of Flexural Modes for a Free Infinite Cylinder (after Zemanek, 1972)
to the family of flexural modes in a plate. There are no analogous modes of propagation for the plate problem corresponding to modes of order higher than the first, i.e. for F(2,m), F(3,m) and above. The flexural family of modes generally involves coupled three dimensional motion $u_r$, $u_\theta$ and $u_z$ displacements (Meeker and Meitzler, 1964).

2.7 SUMMARY

The impact/sonic echo and the impulse response tests that are routinely used for the non-destructive evaluation of deep foundations were described. The assumptions and limitations in these tests were discussed. Two field investigations to evaluate the capability of the non-destructive tests to predict defects in deep foundations were reported. An alternative approach to the simplified one-dimensional assumption of wave propagation is the guided wave approach, which is based on three-dimensional wave propagation. A simplified concept of guided wave propagation in free plates was outlined. The previous work on guided wave propagation in rods was presented.
Chapter 3

Frequency Equation for a Cylindrical Pile Embedded in Soil

3.1 INTRODUCTION

This chapter presents a detailed derivation of the frequency equation that describes wave propagation in an infinitely long solid cylindrical pile embedded in soil. When a pile is excited, waves are assumed to propagate harmonically along the length of the pile and propagate outwards from the center of the pile into the soil. The dynamic equations of motion can be solved by the three-dimensional elasticity approach. The pile and soil are both assumed to be made up of linear elastic, isotropic and homogeneous materials.

The displacements and stresses in the pile and soil are derived in terms of scalar and vector potential functions. The general frequency equation is formulated by applying the boundary conditions. The decomposition of the general frequency equation results in three simpler cases of motion. The case of axisymmetric motion generates the frequency equation for longitudinal modes, represented by a $4 \times 4$ determinant.
3.2 DERIVATION OF THE FREQUENCY EQUATION

3.2.1 General

The derivation of the frequency equation for waves propagating along an infinitely long cylindrical pile embedded in soil is given in this section. The pile and soil are both assumed to be linearly elastic, isotropic and homogeneous materials. Harmonic wave propagation in the pile and soil is assumed and the resulting displacements and stresses in the pile and soil are investigated. A schematic sketch of a cylindrical pile embedded in soil with the cylindrical coordinate axes is shown in Figure 3.1.

![Harmonic wave propagation along pile](image)

Figure 3-1 Schematic Representation of a Cylindrical Pile Embedded in Soil
The displacement equations of motion in isotropic elastic solids in the absence of body forces are given by the Navier-Stokes equation as

\[ \mu_i \nabla^2 \mathbf{u}_i + \left( \lambda_i + \mu_i \right) \nabla \nabla \cdot \mathbf{u}_i = \rho_i \frac{\partial^2 \mathbf{u}_i}{\partial t^2} \]  

(3-1)

where \( \mathbf{u}_i \) is the displacement vector, \( \rho_i \) is the density, \( \lambda_i \) and \( \mu_i \) are the Lamé constants. The subscript \( j \) is used to denote either the pile or the soil; \( p \) is used to represent the pile and \( s \) is used to represent the soil. Letters in boldface denote vectors while those in plain type denote scalars. The symbols \( \nabla \) and \( \nabla^2 \) are the gradient operator and the three dimensional Laplace operator, respectively. In orthogonal cylindrical coordinates, these are given by

\[ \nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{\partial}{\partial z} \hat{z} \]  

(3-2)

and

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \]  

(3-3)

where \( \hat{r}, \hat{\theta} \) and \( \hat{z} \) are the unit vectors in the radial, angular and axial directions respectively.
3.2.2 Boundary Conditions

The pile is assumed to be perfectly bonded to the soil and therefore there must be continuity of displacements and stresses at the pile-soil interface. The solutions to Equation (3-1) must then satisfy the following boundary conditions.

**Displacements**

\[
\begin{align*}
\frac{u_r}{r=a} & = u_r \bigg|_{r=a} \\
\frac{u_\theta}{r=a} & = u_\theta \bigg|_{r=a} \\
\frac{u_z}{r=a} & = u_z \bigg|_{r=a}
\end{align*}
\]  

(3-4)

where \( u_r, u_\theta, u_z \) and \( u_r, u_\theta, u_z \) are the radial, angular and axial displacements in the pile and the soil respectively.

**Stresses**

\[
\begin{align*}
\frac{\sigma_{rr}}{r=a} & = \sigma_{rr} \bigg|_{r=a} \\
\frac{\tau_{r\theta}}{r=a} & = \tau_{r\theta} \bigg|_{r=a} \\
\frac{\tau_{rz}}{r=a} & = \tau_{rz} \bigg|_{r=a}
\end{align*}
\]  

(3-5)

where \( \sigma_{rr}, \tau_{r\theta}, \tau_{rz} \) and \( \sigma_{r\theta}, \tau_{r\theta}, \tau_{rz} \) are the normal and shear stresses in the pile and soil, respectively. The remaining stresses, \( \sigma_{\theta\theta}, \sigma_{zz}, \tau_{z}, \sigma_{\theta z}, \tau_{z\theta} \), are not
interface stresses, and therefore, do not appear in the boundary conditions. It is also assumed that waves propagate harmonically along the axis of the pile and also propagate radially outwards into the soil. Thus, only outward travelling waves are allowed in the soil.

3.2.3 Potentials

The displacement vector may be expressed in terms of a scalar and vector potential in the form

\[ \mathbf{u}_i = \nabla \phi_i + \nabla \times \Psi_i, \]  
(3-6)

\[ \nabla \cdot \Psi_i = F(r, \theta, z, t) \]  
(3-7)

where \( \phi_i \) is the scalar potential, \( \Psi_i \) is the vector potential and \( F(r, \theta, z, t) \) is some function which will be chosen later. Expanding Equation (3-6) and equating terms, the displacement components can then be expressed as

\[ u_r = \frac{\partial \phi_i}{\partial r} + \frac{1}{r} \frac{\partial \psi_z}{\partial \theta} - \frac{\partial \psi_\theta}{\partial z} \]  
(3-8)
\[ u_{\theta} = \frac{1}{r} \frac{\partial \phi_i}{\partial \theta} + \frac{\partial \psi_r}{\partial z} - \frac{\partial \psi_z}{\partial r} \]  
(3-9)

\[ u_z = \frac{\partial \phi_i}{\partial z} + \frac{1}{r} \frac{\partial (r \psi_{\theta_i})}{\partial r} - \frac{1}{r} \frac{\partial \psi_r}{\partial \theta} \]  
(3-10)

The displacement equations of motion are satisfied if the scalar potential \( \phi_i \) and the components of the vector potential, \( \psi_r, \psi_{\theta}, \) and \( \psi_z \) satisfy the scalar and vector wave equations (Graff, 1975). The scalar wave equation is given by Equation (3-11) and the vector wave equations are given by Equations (3-12) to (3-14). In these equations, \( c_k \) and \( c_r \) are the bulk longitudinal and transverse wave velocities, respectively.

\[ \nabla^2 \psi_r = \frac{1}{c_r^2} \frac{\partial^2 \psi_r}{\partial t^2} \]  
(3-11)

\[ \nabla^2 \psi_{\theta} = \frac{\psi_r}{r^2} - \frac{2}{r^2} \frac{\partial \psi_{\theta}}{\partial \theta} = \frac{1}{c_r^2} \frac{\partial^2 \psi_r}{\partial t^2} \]  
(3-12)

\[ \nabla^2 \psi_{\theta} = \frac{\psi_r}{r^2} + \frac{2}{r^2} \frac{\partial \psi_r}{\partial \theta} = \frac{1}{c_r^2} \frac{\partial^2 \psi_{\theta}}{\partial t^2} \]  
(3-13)
\[ \nabla^2 \psi_{z_i} = \frac{c_{r_i}^2}{l^2} \frac{\partial^2 \psi_{z_i}}{\partial t^2} \quad (3-14) \]

The resulting displacements are assumed to propagate harmonically in the steady state. The scalar and vector potentials may then be expressed in general as a function of the radial coordinate, \( r \), and the angular coordinate, \( \theta \), with harmonic plane wave propagation along the axial coordinate, \( z \). The potentials are thus expressed as

\[ \phi_i = f_i(r) \Theta_{r_i} (\theta) e^{i(\omega t - \xi z)} \quad (3-15) \]

\[ \psi_{r_i} = h_{r_i}(r) \Theta_{r_i} (\theta) e^{i(\omega t - \xi z)} \quad (3-16) \]

\[ \psi_{\theta_i} = h_{\theta_i}(r) \Theta_{\theta_i} (\theta) e^{i(\omega t - \xi z)} \quad (3-17) \]

\[ \psi_{z_i} = h_{z_i}(r) \Theta_{z_i} (\theta) e^{i(\omega t - \xi z)} \quad (3-18) \]

where \( f_i \), \( h_{r_i} \), \( h_{\theta_i} \), and \( h_{z_i} \) are arbitrary functions of the radial coordinate, \( \Theta_{r_i}, \Theta_{\theta_i} \) and \( \Theta_{z_i} \) are arbitrary functions of the angular coordinate, \( \omega \) is the angular frequency and \( \xi \) is the wave number. The forms of these arbitrary functions are determined by substituting them into the scalar and vector wave equations.
Substituting Equation (3-15) into (3-11) gives

\[
\frac{r^2}{f_i} \frac{\partial^2 f_i}{\partial r^2} + \frac{r}{f_i} \frac{\partial f_i}{\partial r} + \frac{\partial^2 \Theta_{\phi}}{\partial \Theta_{\phi}^2} + \left( \frac{\omega^2}{c_{t_i}} - \xi^2 \right) r^2 = 0
\]  
\tag{3-19}

Introducing the following substitution

\[
\alpha_i^2 = \frac{\omega^2}{c_{t_i}^2} - \xi^2
\]  
\tag{3-20}

Substituting Equation (3-20) into (3-19) gives

\[
\frac{r^2}{f_i} \frac{\partial^2 f_i}{\partial r^2} + \frac{r}{f_i} \frac{\partial f_i}{\partial r} + \frac{\partial^2 \Theta_{\phi}}{\partial \Theta_{\phi}^2} + \alpha_i^2 r^2 = 0
\]  
\tag{3-21}

Using separation of variables, Equation (3-21) can be arranged into the following form

\[
\frac{r^2}{f_i} \frac{\partial^2 f_i}{\partial r^2} + \frac{r}{f_i} \frac{\partial f_i}{\partial r} + \alpha_i^2 r^2 = -\frac{\partial^2 \Theta_{\phi}}{\partial \Theta_{\phi}^2} = n^2
\]  
\tag{3-22}

where \( n \) is a constant.
The expressions in Equation (3-22) may be separated into the following two equations

\[
\frac{\partial^2 \Theta_{\phi}}{\partial \Theta_{\phi}^2} + n^2 \Theta_{\phi}^2 = 0 \tag{3-23}
\]

\[
r^2 \frac{\partial^2 f_i}{\partial r^2} + r \frac{\partial f_i}{\partial r} + \left( \alpha_i^2 r^2 - n^2 \right) f_i = 0 \tag{3-24}
\]

Equation (3-23) is a second order partial differential equation with constant coefficients while Equation (3-24) is a Bessel equation. The general solution to Equation (3-23) is of the form

\[
\Theta_{\phi} = C_i \cos n \theta + C_{k_i} \sin n \theta, \quad n = 0, 1, 2, 3, \ldots \tag{3-25}
\]

where \( C_i \) and \( C_{k_i} \) are constants.

The boundary condition for the \( \Theta_{\phi} \) function is periodicity in \( \theta \) and therefore the solution for \( \Theta_{\phi} \) should be continuous functions of \( \theta \) with continuous derivatives. This requirement leads to \( n \) being an integer. Likewise, substituting the potentials shown in Equations (3-16), (3-17) and (3-18) into the vector wave equations shown in Equations (3-12), (3-13) and (3-14), result in similar solutions for \( \Theta_{r_i}, \Theta_{\phi_i}, \) and \( \Theta_{z_i} \).
The nature of the $\theta$ dependence for the longitudinal, torsional and flexural modes would lead to discarding either the sine or cosine term in the $\Theta_\phi, \Theta_r, \Theta_\theta$, and $\Theta_z$, expressions (Graff, 1975). Accordingly, the expressions for the potentials become

$$\phi_j = f_j(r) \cos n\theta \ e^{(a_j - \omega t)}$$  \hspace{1cm} (3-26)

$$\psi_{r_j} = h_{r_j}(r) \sin n\theta \ e^{(a_j - \omega t)}$$  \hspace{1cm} (3-27)

$$\psi_{\theta_j} = h_{\theta_j}(r) \cos n\theta \ e^{(a_j - \omega t)}$$  \hspace{1cm} (3-28)

$$\psi_{z_j} = h_{z_j}(r) \sin n\theta \ e^{(a_j - \omega t)}$$  \hspace{1cm} (3-29)

The general solution to the Bessel equation in Equation (3-24) is

$$f_j(r) = A_j Z_n(\alpha_j r) + B_j W_n(\alpha_j r)$$  \hspace{1cm} (3-30)

where $A_j$ and $B_j$ are constants; and $\alpha_j$ is the magnitude of $\alpha_j$. For real values of $\alpha_j$, $Z_n$ and $W_n$ represent the ordinary Bessel functions $J_n$ and $Y_n$, respectively, while
for imaginary values of $\alpha$, $Z_n$ and $W_n$ represent the modified Bessel functions $I_n$ and $K_n$ (Gazis, 1959).

Substituting the potentials shown in Equations (3-27) and (3-28) into the vector wave equations in Equations (3-12) and (3-13) gives

\[
\frac{\partial^2 h_r}{\partial r^2} + \frac{1}{r} \frac{\partial h_r}{\partial r} + \frac{1}{r^2} \left( -n^2 h_r + 2nh_\theta - h_r \right) - \xi^2 h_r + \frac{\omega^2}{c_r^2} h_r = 0 \tag{3-31}
\]

\[
\frac{\partial^2 h_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial h_\theta}{\partial r} + \frac{1}{r^2} \left( -n^2 h_\theta + 2nh_r - h_\theta \right) - \xi^2 h_\theta + \frac{\omega^2}{c_r^2} h_\theta = 0 \tag{3-32}
\]

Subtracting Equation (3-32) from Equation (3-31) gives

\[
\frac{\partial^2 (h_r - h_\theta)}{\partial r^2} + \frac{1}{r} \frac{\partial (h_r - h_\theta)}{\partial r} + \left[ \frac{\omega^2}{c_r^2} - \xi^2 \right] \left( h_r - h_\theta \right) = 0 \tag{3-33}
\]

Introducing the following substitution

\[
\beta_j^2 = \frac{\omega^2}{c_r^2} - \xi^2 \tag{3-34}
\]
Substituting Equation (3-34) into (3-33) gives

\[
\frac{\partial^2 (h_r - h_\theta)}{\partial r^2} + \frac{1}{r} \frac{\partial (h_r - h_\theta)}{\partial r} + \left( \beta_i^2 - \frac{(n+1)^2}{r^2} \right) (h_r - h_\theta) = 0
\]  

(3-35)

The general solution to the Bessel equation in Equation (3-35) is

\[
h_r - h_\theta = A_i Z_{n,i}(\beta_i r) + B_i W_{n,i}(\beta_i r)
\]

(3-36)

where \(A_i\) and \(B_i\) are constants and \(\beta_i\) is the magnitude of \(\beta_i\). For real values of \(\beta_i\), \(Z_n\) and \(W_n\) represent the ordinary Bessel functions \(J_n\) and \(Y_n\), respectively, while for imaginary values of \(\beta_i\), \(Z_n\) and \(W_n\) represent the modified Bessel functions \(I_n\) and \(K_n\).

Adding Equation (3-32) to Equation (3-31) gives

\[
\frac{\partial^2 (h_r + h_\theta)}{\partial r^2} + \frac{1}{r} \frac{\partial (h_r + h_\theta)}{\partial r} + \left( \beta_i^2 - \frac{(n-1)^2}{r^2} \right) (h_r + h_\theta) = 0
\]

(3-37)
The general solution to the Bessel equation in Equation (3-37) is

\[ h_r + h_\theta = A_j Z_{n-1}(\beta_j r) + B_j W_{n-1}(\beta_j r) \]  \hspace{1cm} (3-38)

where \( A_j \) and \( B_j \) are constants. Two of the integration constants appearing in the above equations can be eliminated by specifying the condition for the function \( F(r, \theta, z, t) \). Any one of the equivoluminal potentials can be set to zero without loss of generality of the solution. This means that the displacement field corresponding to an equivoluminal potential can also be derived by a combination of the other two equivoluminal potentials (Gazis, 1959). Thus, setting \( A_j = 0 \) and \( B_j = 0 \) gives

\[ h_r = -h_\theta, \]  \hspace{1cm} (3-39)

Substituting Equation (3-39) into (3-36) gives

\[ h_r = A_j Z_{n-1}(\beta_j r) + B_j W_{n-1}(\beta_j r) \]  \hspace{1cm} (3-40)

where \( A_j = \frac{A_{2j}}{2} \) and \( B_j = \frac{B_{2j}}{2} \).
Finally substituting Equation (3-29) into (3-14) gives

\[
 r^2 \frac{\partial^2 h_z}{\partial r^2} + r \frac{\partial h_z}{\partial r} + \left( \beta_i^2 r^2 - n^2 \right) h_z = 0 \quad (3-41)
\]

The general solution to the Bessel equation in Equation (3-41) is given by

\[
h_{z_i}(r) = A_i Z_{n_i}(\beta_i r) + B_i W_{n_i}(\beta_i r) \quad (3-42)
\]

where \( A_i \) and \( B_i \) are constants.

### 3.2.4 Selection of Radial Functions in the Scalar and Vector Potentials

The general form of the radial functions in the scalar and vector potentials have
been determined in the previous section, and are given by Equations (3-30), (3-39),
(3-40) and (3-42). Because of the equality in Equation (3-39), only six radial functions
appear in the scalar and vector potentials, three for the pile and three for the soil. The
particular form of the radial functions will depend on the geometry of the problem and
the boundary conditions.

Consider the radial functions for the pile given by Equations (3-30), (3-40) and
(3-42). Because the pile is solid, these functions must be finite valued at the center of
the pile where \( r=0 \). The Bessel functions \( Y \) and \( K \) both are asymptotic to infinity at \( r=0 \) and therefore are unsuitable. Thus, the radial functions for the pile are given by

\[
f_r = A_r Z_n(\bar{\alpha}_r r)
\]  
(3-43)

\[
h_r = A_n Z_{n+1}(\bar{\beta}_r r)
\]  
(3-44)

\[
h_{r,s} = A_n Z_n(\bar{\beta}_r r)
\]  
(3-45)

Similarly, appropriate radial functions are required for the soil. As mentioned in Section 3.2.2, the assumed harmonic wave propagation in the pile results in the wave energy propagating radially outwards into the soil. Therefore the functions should represent outgoing waves which decay with distance in the radial direction, as the boundaries are located at infinity. The ordinary or modified Bessel functions on their own cannot describe this behavior. However, they may be combined to form the Hankel function, which describes the required behavior. Hence, the radial functions in the soil are given by

\[
f_s = A_i H_n^{(2)}(\bar{\alpha},r)
\]  
(3-46)
\[ h_r = A_4 \cdot H_n^{(2)}(\bar{\beta}, r) \quad (3-47) \]

\[ h_z = A_5 \cdot H_n^{(2)}(\bar{\beta}, r) \quad (3-48) \]

where \( H_n^{(2)} \) is the Hankel function of the second kind and is given by

\[ H_n^{(2)}(x) = J_n(x) - iY_n(x) \quad (3-49) \]

The asymptotic behavior of the real part of the Hankel function, \( \text{Re}[H_n^{(2)}(x)] \) with large arguments is illustrated in Figure 3-2.

![Figure 3-2 Asymptotic Behavior of the Hankel Function](image)
3.2.5 Displacements

The displacements may now be expressed in terms of the potential functions.

Substituting the partial derivatives of Equations (3-26), (3-27), (3-28) and (3-29) into Equations (3-8), (3-9) and (3-10) gives

\[ u_r = \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial \psi_{z_i}}{\partial \theta} - \frac{\partial \psi_{\theta_i}}{\partial z} \]

\[ = f_i \cos n \theta e^{i(\omega r - \xi z)} + \frac{n}{r} h_i \cos n \theta e^{i(\omega r - \xi z)} + i \xi h_{\theta_i} \cos n \theta e^{i(\omega r - \xi z)} \]  \hspace{1cm} (3-50)

\[ = \left[ f_i + \frac{n}{r} h_i, -i \xi h_{\theta_i} \right] \cos n \theta e^{i(\omega r - \xi z)} \]

\[ u_{\theta_i} = \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \frac{\partial \psi_{z_i}}{\partial z} - \frac{\partial \psi_{\theta_i}}{\partial \theta} \]

\[ = -n \frac{f_i}{r} \sin n \theta e^{i(\omega r - \xi z)} - i \xi h_{\theta_i} \sin n \theta e^{i(\omega r - \xi z)} - h_{z_i} \sin n \theta e^{i(\omega r - \xi z)} \]  \hspace{1cm} (3-51)

\[ = \left[ -n \frac{f_i}{r} - i \xi h_{\theta_i} - h_{z_i} \right] \sin n \theta e^{i(\omega r - \xi z)} \]

\[ u_{z_i} = \frac{\partial \phi}{\partial z} + \frac{1}{r} \frac{\partial (r \psi_{\theta_i})}{\partial r} \frac{1}{r} \frac{\partial \psi_{r_i}}{\partial \theta} \]

\[ = -i \xi f_i \cos n \theta e^{i(\omega r - \xi z)} + h_{\theta_i} \cos n \theta e^{i(\omega r - \xi z)} + \frac{h_{\theta_i}}{r} \cos n \theta e^{i(\omega r - \xi z)} \]

\[ - \frac{n}{r} h_{r_i} \cos n \theta e^{i(\omega r - \xi z)} \]  \hspace{1cm} (3-52)

\[ = \left[ -i \xi f_i - h_{r_i} - \frac{h_{r_i}}{r} (n + l) \right] \cos n \theta e^{i(\omega r - \xi z)} \]
where single primes in $f', h'_r, h'_p$ and $h'_p$ denote first derivatives with respect to $r$. For convenience, replace $-ih_r$ by $h_r$ and $-ih'_r$ by $h'_r$ in Equations (3-50), (3-51) and (3-52) by absorbing $-i$ into the coefficients $A_i$ and $B_i$ of Equation (3-40). The displacements can then be expressed as

$$u_r = \left[ f'_r + \frac{n}{r} h'_r + \xi h_r \right] \cos n\theta \ e^{i(\omega t-\xi z)}$$  \hspace{1cm} (3-53)

$$u_{\theta} = \left[ -\frac{n}{r} f'_r + \xi h'_r - h'_p \right] \sin n\theta \ e^{i(\omega t-\xi z)}$$  \hspace{1cm} (3-54)

$$u_z = i \left[ -\xi f'_r - h'_r - \frac{h_p}{r} (n + l) \right] \cos n\theta \ e^{i(\omega t-\xi z)}$$  \hspace{1cm} (3-55)

The displacements in the pile are expressed in terms of ordinary and modified Bessel functions of the first kind by substituting Equations (3-43), (3-44) and (3-45) into Equations (3-53), (3-54) and (3-55), which gives

$$u_r = \left[ f'_r + \frac{n}{r} h'_r + \xi h_r \right] \cos n\theta \ e^{i(\omega t-\xi z)}$$

$$= \left[ A_i \bar{\alpha}_p Z'_n(\bar{\alpha}_p r) + A_i \bar{\alpha}_r Z_n(\bar{\beta}_p r) + A_i \xi Z_{n+1}(\bar{\beta}_p r) \right] \cos n\theta \ e^{i(\omega t-\xi z)}$$  \hspace{1cm} (3-56)
\[ u_{\theta_p} = \left[ -\frac{n}{r} f_p + \xi h_p - h'_p \right] \sin n\theta \, e^{i(\omega r - \xi z)} = \left[ -A_{t_1} \frac{n}{r} Z_n(\xi r) + A_{t_2} \xi Z_{n+1}(\xi r) - A_{t_3} \xi Z_n' \right] \sin n\theta \, e^{i(\omega r - \xi z)} \]  

(3-57)

\[ u_{z_p} = i \left[ -\frac{\xi f_p - h'_p - \frac{(n+1)}{r} h_p}{r} \right] \cos n\theta \, e^{i(\omega r - \xi z)} = i \left[ -A_{t_1} \xi Z_n(\xi r) - A_{t_2} \xi Z_{n+1}(\xi r) - A_{t_3} \frac{(n+1)}{r} Z_n' \right] \cos n\theta \, e^{i(\omega r - \xi z)} \]  

(3-58)

It is convenient to recall the following Bessel function identities

\[ Z_n^*(x) = \left( \frac{n^2}{x^2} - 1 \right) Z_n(x) - \frac{i}{x} Z_n'(x) \]  

(3-59)

\[ Z_{n+1}(x) = Z_n(x) - \frac{(n+1)}{x} Z_n'(x) \]  

(3-60)

\[ Z_{n+1}(x) = \frac{n}{x} Z_n(x) - Z_n'(x) \]  

(3-61)

where the single and double primes indicate first and second derivatives of the Bessel function with respect to its argument. Substituting Equations (3-60) and (3-61) into Equations (3-56), (3-57) and (3-58) and simplifying gives
\[ u_{r} = \left\{ A_{r} \alpha_{p} Z_{n}^{'}(\alpha_{p} r) - A_{r} \alpha_{p} Z_{n}^{'}(\beta_{p} r) + \left[ A_{s} \xi \beta_{p} + A_{s} \right] \frac{n}{r} Z_{n}(\beta_{p} r) \right\} \cos n\theta \, e^{i(\omega t - \xi)} \tag{3-62} \]

\[ u_{\theta} = \left\{ -A_{r} \frac{n}{r} Z_{n}(\alpha_{p} r) + A_{r} \frac{n}{r} Z_{n}(\beta_{p} r) - \left[ A_{s} \xi + A_{s} \beta_{p} \right] Z_{n}(\beta_{p} r) \right\} \sin n\theta \, e^{i(\omega t - \xi)} \tag{3-63} \]

\[ u_{r} = i \left[ -A_{r} \xi Z_{n}(\alpha_{p} r) - A_{s} \beta_{p} Z_{n}(\beta_{p} r) \right] \cos n\theta \, e^{i(\omega t - \xi)} \tag{3-64} \]

Equations (3-62) and (3-63) may be further simplified using the following substitution

\[ A_{s} = \frac{A_{s} \xi}{\beta_{p}} + A_{s} \tag{3-65} \]

Substituting Equation (3-65) into Equations (3-62) and (3-63) gives

\[ u_{r} = \left[ A_{r} \alpha_{p} Z_{n}^{'}(\alpha_{p} r) - A_{r} \alpha_{p} Z_{n}^{'}(\beta_{p} r) + A_{s} \frac{n}{r} Z_{n}(\beta_{p} r) \right] \cos n\theta \, e^{i(\omega t - \xi)} \tag{3-66} \]

\[ u_{\theta} = \left[ -A_{s} \frac{n}{r} Z_{n}(\alpha_{p} r) + A_{r} \frac{n}{r} Z_{n}(\beta_{p} r) - A_{s} \beta_{p} Z_{n}(\beta_{p} r) \right] \sin n\theta \, e^{i(\omega t - \xi)} \tag{3-67} \]
Similarly, expressions for displacements in the soil can be developed to give

\[ u_r = \left[ A_1, \alpha_n H^{(2)}(\alpha, r) - A_4, \xi H^{(2)}(\beta, r) + A_n, \frac{n}{r} H^{(2)}(\beta, r) \right] \cos n\theta e^{i(\omega t - \xi z)} \] (3-68)

\[ u_\theta = \left[ -A_1, \frac{n}{r} H^{(2)}(\alpha, r) + A_4, \frac{n}{\beta, r} H^{(2)}(\beta, r) - A_n, H^{(2)}(\beta, r) \right] \sin n\theta e^{i(\omega t - \xi z)} \] (3-69)

\[ u_z = i \left[ -A_1, \xi H^{(2)}(\alpha, r) - A_4, \beta, H^{(2)}(\beta, r) \right] \cos n\theta e^{i(\omega t - \xi z)} \] (3-70)

3.2.6 Strain-Displacement Relations

For the axisymmetric conditions considered herein, the radial, tangential and axial normal strains are given by \( \varepsilon_{rr}, \varepsilon_{\theta\theta}, \) and \( \varepsilon_z, \) respectively. These are given in cylindrical coordinates by

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} \] (3-71)

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \left( u_r + \frac{\partial u_\theta}{\partial \theta} \right) \] (3-72)
\[ \varepsilon_{z} = \frac{\partial u_z}{\partial z} \]  
(3-73)

The non-zero shear strain components in cylindrical coordinates are given by

\[ \varepsilon_{r\theta} = \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{u_\theta}{r} \right) \right] \]  
(3-74)

\[ \varepsilon_{\rho z} = \frac{1}{2} \left[ \frac{\partial u_\rho}{\partial z} + \frac{\partial u_z}{\partial r} \right] \]  
(3-75)

Assuming small strains, the dilatation, \( \Delta_j \), is given by,

\[ \Delta_j = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} \]  
(3-76)

Substituting Equations (3-71), (3-72) and (3-73) into Equation (3-76) gives

\[ \Delta_j = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \]  
(3-77)
Substituting Equations (3-8), (3-9) and (3-10) into Equation (3-77) and simplifying gives

\[
\Delta_j = \frac{\partial^2 \phi_j}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_j}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi_j}{\partial \theta^2} + \frac{\partial^2 \phi_j}{\partial z^2}
\]

\[
= \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \right) \phi_j
\]

\[
= \nabla^2 \phi_j
\]

### 3.2.7 Stress-Strain Relations

The radial, tangential and axial normal stresses are given by \( \sigma_{rr} \), \( \sigma_{\theta\theta} \), and \( \sigma_{zz} \), respectively. These are expressed in cylindrical coordinates as

\[
\sigma_{rr} = \lambda_j \Delta_j + 2\mu_j \frac{\partial u_r}{\partial r}
\]

\[
\sigma_{\theta\theta} = \lambda_j \Delta_j + 2\mu_j \left( \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_{\theta}}{\partial \theta} \right)
\]

\[
\sigma_{zz} = \lambda_j \Delta_j + 2\mu_j \frac{\partial u_z}{\partial z}
\]
The non-zero shear stresses are expressed as

\[ \tau_{r\theta} = \mu_1 \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_{\theta}}{\partial r} - \frac{u_\theta}{r} \right) \]  

(3-82)

\[ \tau_{r\zeta} = \mu_1 \left[ \frac{\partial u_r}{\partial \zeta} + \frac{\partial u_z}{\partial r} \right] \]  

(3-83)

3.2.8 Stresses

The stresses can now be expressed in terms of the potential functions. The expressions for \( \sigma_{\theta \theta}, \sigma_z \) and \( \tau_{\theta z} \) are not evaluated as these stresses are not used in subsequent computations.

Radial Stress \( (\sigma_{rr}) \)

Substituting Equation (3-11) into Equation (3-78) and taking the derivatives with respect to time and using Equation (3-26) gives

\[ \Delta_j = \frac{l}{c_{t_j}^2} \frac{\partial^2 \phi_j}{\partial t^2} \]

(3-84)

\[ = -\frac{\omega^2}{c_{t_j}^2} f_j(r) \cos \theta e^{i(\omega r - \zeta)} \]
Partially differentiating Equation (3-53) with respect to $r$ gives

$$\frac{\partial u_{r_1}}{\partial r} = \left[ f_i^{^*} + \frac{n}{r} h_i^{^*} - \frac{n}{r^2} h_i + \xi h_i^{^*} \right] \cos n\theta e^{i(\omega t - \xi z)} \quad (3-85)$$

Substituting Equation (3-20), (3-84) and (3-85) into Equation (3-79) gives

$$\sigma_{r_1} = \left\{ -\lambda_i \left( \alpha_i + \xi \right) f_i + 2\mu_i \left[ f_i^{^*} + \frac{n}{r} h_i^{^*} - \frac{n}{r} h_i + \xi h_i^{^*} \right] \right\} \cos n\theta e^{i(\omega t - \xi z)} \quad (3-86)$$

**Shear Stress ($\tau_{r\theta}$)**

Expanding the components of $\tau_{r\theta}$, shown in Equation (3-82) gives

$$\frac{i}{r} \frac{\partial u_{r_1}}{\partial \theta} = -\frac{n}{r} \left[ f_i^{^*} + \frac{n}{r} h_i^{^*} + \xi h_i^{^*} \right] \sin n\theta e^{i(\omega t - \xi z)} \quad (3-87)$$

and

$$\frac{\partial u_{r_1}}{\partial r} = \left[ -\frac{n}{r^2} f_i + \frac{n}{r^2} f_i^{^*} + \xi h_i^{^*} - h_i^{^*} \right] \sin n\theta e^{i(\omega t - \xi z)} \quad (3-88)$$

and

$$\frac{u_{r_1}}{r} = \left[ -\frac{n}{r^2} f_i + \frac{n}{r} h_i^{^*} - \frac{l}{r} h_i^{^*} \right] \sin n\theta e^{i(\omega t - \xi z)} \quad (3-89)$$
Substituting Equations (3-87), (3-88) and (3-89) into Equation (3-82) gives

\[
\tau_{r\theta_i} = \mu_i \left\{ \frac{2n}{r} \left( f'_i - f_\perp \right) - h''_{z_i} + \frac{l}{r} h'_{z_i} \right\} \sin n\theta \ e^{(\omega t - \xi z)}
\]

\[
-\frac{n^2}{r^2} h_{z_i} - \frac{\xi}{r} \left[ (n + l) h_{r_i} - rh'_{r_i} \right]
\]

(3-90)

Equation (3-41) can be rearranged as

\[
\frac{l}{r} h'_{z_i} = -h''_{z_i} - \left( \beta^2_i - \frac{n^2}{r^2} \right) h_{z_i}
\]

(3-91)

The expression for \( \tau_{r\theta_i} \) is simplified by substituting Equation (3-91) into Equation (3-90) to give

\[
\tau_{r\theta_i} = \mu_i \left\{ \frac{2n}{r} \left( f'_i - f_\perp \right) - \left( 2h''_{z_i} + \beta^2_i h_{z_i} \right) - \frac{\xi}{r} \left[ \left( \frac{n + l}{r} \right) h_{r_i} - h'_{r_i} \right] \right\} \sin n\theta \ e^{(\omega t - \xi z)}
\]

(3-92)

**Shear Stress** (\( \tau_{r_i} \))

Expanding the components of \( \tau_{r_i} \), shown in Equation (3-83) gives
\[ \frac{\partial u_r}{\partial z} = -i \xi \left[ f'_r + \frac{n}{r} h'_r + \xi h_r \right] \cos \theta e^{(\omega r - \xi z)} \] \tag{3-93} 

and

\[ \frac{\partial u_z}{\partial r} = \left[ -\xi f'_r - h''_r - \frac{(n+1)}{r} h'_r + \frac{(n+1)}{r^2} h_r \right] \cos \theta e^{(\omega r - \xi z)} \] \tag{3-94} 

Substituting Equations (3-93) and (3-94) into Equation (3-83) gives

\[ \tau_n = i \mu_j \left\{ -2 \xi f'_r - \frac{n \xi}{r} h'_r - h''_r - \frac{(n+1)}{r} h'_r + \left[ \frac{(n+1)^2}{r^2} - \xi^2 \right] h_r \right\} \cos \theta e^{(\omega r - \xi z)} \] \tag{3-95} 

Substitute Equations (3-34) and (3-39) into Equation (3-31) to give

\[ h''_r = -h'_r + \frac{(n+1)^2}{r^2} - \beta^2 \] \tag{3-96} 

The expression for \( \tau_n \) is simplified by substituting Equation (3-96) into Equation (3-95) and rearranging to give

\[ \tau_n = i \mu_j \left\{ -2 \xi f'_r - \frac{n \xi}{r} h'_r - h''_r - \left[ \xi^2 - \beta^2 \right] + \frac{n(n+1)}{r} \right\} h_r \cos \theta e^{(\omega r - \xi z)} \] \tag{3-97}
The stresses in the pile are expressed in terms of ordinary and modified Bessel functions of the first kind by substituting Equations (3-43), (3-44) and (3-45) and their derivatives, successively into Equations (3-86), (3-92) and (3-97). Rearranging these expressions gives

\[
\sigma_{rr} = \left\{ \begin{array}{c}
A_i \left[ -\lambda_p \left( \alpha_p^2 + \xi^2 \right) Z_n(\alpha_p r) + 2 \mu_p \alpha_p^2 Z_n^*(\alpha_p r) \right] \\
+ A_i \left[ 2 \mu_p A_p^2 Z_{n+1}(\alpha_p r) \right] \\
+ A_i \left[ 2 \mu_p \frac{n A_p^2}{r} Z_n(\alpha_p r) - 2 \mu_p \frac{n}{r^2} Z_n^*(\alpha_p r) \right]
\end{array} \right\} \cos \theta \ e^{i(\omega_1 - \xi_2)}
\] (3-98)

\[
\tau_{r\theta} = \mu_p \left\{ \begin{array}{c}
A_i \left[ \frac{2n}{r} \left( \frac{1}{r} Z_n(\alpha_p r) - \alpha_p Z_n^*(\alpha_p r) \right) \right] \\
+ A_i \xi \left[ \beta_p Z_{n+1}(\alpha_p r) - \left( \frac{n + l}{r} \right) Z_n(\alpha_p r) \right] \\
+ A_i \left[ -2 \beta_p^2 Z_{n+1}(\alpha_p r) + 2 \beta_p^2 Z_n(\alpha_p r) \right]
\end{array} \right\} \sin \theta \ e^{i(\omega_1 - \xi_2)}
\] (3-99)

\[
\tau_{\theta r} = i \mu_p \left\{ \begin{array}{c}
A_i \left[ -2 \xi \alpha_p Z_n^*(\alpha_p r) \right] \\
A_i \left[ \frac{n A_p^2}{r} Z_{n+1}(\alpha_p r) \right] \\
- \left( \alpha^2 - \beta^2 \right) + \left( \frac{n + l}{r} \right) Z_{n+1}(\alpha_p r) \\
A_i \left[ - \frac{n A_p^2}{r} Z_n(\alpha_p r) \right]
\end{array} \right\} \cos \theta \ e^{i(\omega_1 - \xi_2)}
\] (3-100)
Using Bessel function identities, substituting Equations (3-59), (3-60) and (3-61) successively into Equation (3-98), (3-99) and (3-100), and rearranging gives

\[
\sigma_{rr} = \left\{ \begin{array}{l}
A_{lr} \left[ -\lambda_p \left( \frac{\alpha_r^2}{r^2} + \xi^2 \right) + 2\mu_p \left( \frac{n^2}{r^2} - \frac{\alpha_r^2}{r^2} \right) \right] Z_n(\alpha_r r)
-2\mu_p \frac{\alpha_r}{r} Z'(\alpha_r r)
\end{array} \right\} + A_{lr} 2\mu_p \xi \left[ \frac{2}{r} \right] \left[ \frac{n}{\beta_r} \right] Z_n(\beta_r r) - Z'(\beta_r r) \right\} + \cos n\theta e^{i(n-\xi z)}
\]

(3-101)

\[
\tau_{\theta r} = \mu_p \left\{ \begin{array}{l}
A_{lr} \frac{2n}{r} \left[ \frac{1}{r} Z_n(\alpha_r r) - \alpha_r Z'(\alpha_r r) \right]
A_{lr} \xi \left[ \frac{2n(n+1)}{r} \right] Z_n(\beta_r r) + \frac{2(n+1)}{r} Z'(\beta_r r) \right\} + \sin n\theta e^{i(n-\xi z)}
\right\}
\]

(3-102)

\[
\tau_{\phi r} = i\mu_p \left\{ \begin{array}{l}
A_{lr} \left[ -2\xi \alpha_r Z'(\alpha_r r) \right]
- \frac{n^2}{\beta_r} Z_n(\beta_r r) + (\xi^2 - \beta_r^2) Z'(\beta_r r) \right\} + \cos n\theta e^{i(n-\xi z)}
\right\}
\]

(3-103)
Using the substitution shown in Equation (3-65) in Equations (3-101), (3-102) and (3-103) successively gives

\[
\sigma_{rr} = \begin{pmatrix}
-\lambda_p \left( \bar{\alpha}_p^2 + \xi^2 \right) + 2 \mu_p \left( \frac{n^2}{r^2} - \bar{\alpha}_p^2 \right) Z_n(\bar{\alpha}_p r)
- 2 \mu_p \frac{\bar{\alpha}_p}{r} Z'_n(\bar{\alpha}_p r)
+ A_t \nu \bar{\alpha}_p Z_n(\bar{\alpha}_p r) + \frac{1}{r} Z'_n(\bar{\alpha}_p r)
+ A_s \nu \bar{\alpha}_p Z_n(\bar{\alpha}_p r) - \frac{1}{r} Z'_n(\bar{\alpha}_p r)
\end{pmatrix} \cos n\theta e^{i(\omega t - \xi)}
\]

(3-104)

\[
\tau_{r\theta} = \begin{pmatrix}
A_s \frac{2n}{r} \left[ \frac{l}{r} Z_n(\bar{\alpha}_p r) - \bar{\alpha}_p Z'_n(\bar{\alpha}_p r) \right]
+ A_s \bar{\alpha}_p Z_n(\bar{\alpha}_p r) - \frac{2n^2}{r^2} Z_n(\bar{\alpha}_p r)
\end{pmatrix} \sin n\theta e^{i(\omega t - \xi)}
\]

(3-105)

\[
\tau_{r\phi} = i \mu_p \left[ -2 \xi \bar{\alpha}_p Z_n(\bar{\alpha}_p r) \right] + A_s \left( \xi^2 - \bar{\alpha}_p^2 \right) Z'_n(\bar{\alpha}_p r) + \frac{n^2}{r} Z_n(\bar{\alpha}_p r) \cos n\theta e^{i(\omega t - \xi)}
\]

(3-106)
Similarly, developing the expressions for stresses in the soil gives

\[
\sigma_{m} = \left\{ \begin{array}{l}
A_{1} \left\{ - \frac{2}{r} \left( \alpha_{r}^{2} + \alpha_{\theta}^{2} \right) + 2 \mu_{s} \left( \frac{n^{2}}{r^{2}} - \alpha_{r}^{2} \right) \right\} H_{n}^{(2)}(\alpha, r) \\
- 2 \mu_{s} \frac{\alpha_{r}}{r} H_{n}^{(2)}(\alpha, r) \\
+ A_{4} 2 \mu_{s} \left( \frac{1}{r} \left( \bar{\beta}_{r}^{2} - \frac{n^{2}}{\bar{\beta}_{\theta}^{2}} \right) H_{n}^{(2)}(\bar{\beta}, r) - \frac{1}{r} H_{n}^{(2)}(\bar{\beta}, r) \right)
\end{array} \right\} \cos \theta e^{i(\omega t - \xi z)} \tag{3-107}
\]

\[
\tau_{\rho \theta} = \mu_{s} \left\{ \begin{array}{l}
A_{1} \frac{2n^{2}}{r} \left[ \frac{1}{r} H_{n}^{(2)}(\alpha, r) - \alpha_{r} H_{n}^{(2)}(\alpha, r) \right] \\
+ A_{4} \frac{2n^{2}}{r} \left[ \frac{1}{r} H_{n}^{(2)}(\bar{\beta}, r) - \frac{1}{\bar{\beta}_{r}} H_{n}^{(2)}(\bar{\beta}, r) \right]
\end{array} \right\} \sin \theta e^{i(\omega t - \xi z)} \tag{3-108}
\]

\[
\tau_{\rho \phi} = i \mu_{s} \left\{ \begin{array}{l}
A_{1} \left[ - 2 \xi \alpha_{r} H_{n}^{(2)}(\alpha, r) \right] \\
+ A_{4} \left[ \bar{\beta}_{r}^{2} \right] H_{n}^{(2)}(\bar{\beta}, r) \right\} \cos \theta e^{i(\omega t - \xi z)} \tag{3-109}
\]

\[
A_{1} = \frac{1}{2} \left( \frac{1}{r_{0}} \right)^{2} \left( \frac{1}{r_{0}} \right)^{2} - \frac{1}{r_{0}} \left( \frac{1}{r_{0}} \right)^{2} \right)
\]

\[
A_{4} = \frac{1}{2} \left( \frac{1}{r_{0}} \right)^{2} \left( \frac{1}{r_{0}} \right)^{2} - \frac{1}{r_{0}} \left( \frac{1}{r_{0}} \right)^{2} \right)
\]

\[
A_{5} = \frac{1}{2} \left( \frac{1}{r_{0}} \right)^{2} \left( \frac{1}{r_{0}} \right)^{2} - \frac{1}{r_{0}} \left( \frac{1}{r_{0}} \right)^{2} \right)
\]

\[
A_{6} = \frac{1}{2} \left( \frac{1}{r_{0}} \right)^{2} \left( \frac{1}{r_{0}} \right)^{2} - \frac{1}{r_{0}} \left( \frac{1}{r_{0}} \right)^{2} \right)
\]
3.2.9 Matrix Formulation

The displacement and stress boundary conditions, shown in Equations (3-4) and (3-5), may be written in matrix notation as

\[
\begin{pmatrix}
u_{r, p} - u_r,
\varepsilon_{r, r, p} - \varepsilon_{r, r},
\gamma_{r, r, p} - \gamma_{r, r},
\tau_{r, r, p} - \tau_{r, r}
\end{pmatrix} = 0
\]  
(3.110)

Substituting the displacements and stresses derived in Sections 3.2.5 and 3.2.8 into Equation (3-110), and rearranging yields

\[
\begin{pmatrix}
x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16}
x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26}
x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36}
x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46}
x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56}
x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{pmatrix}
\begin{pmatrix}
A_{x, r}
A_{y, r}
A_{z, r}
A_{s, r}
A_{t, r}
A_{d, r}
\end{pmatrix} = 0
\]  
(3-111)

where the \( x_{nn} \) terms in the coefficient matrix are given in Equations (3-112) to (3-117). The unknown constants are shown in the unknown matrix. Equation (3-111)
is a linear system of equations in the unknowns. Because the system of equations is homogeneous, it may have either the trivial solution or an infinite number of solutions. The non-trivial solutions are obtained by setting the determinant of the coefficient matrix equal to zero. This results in a frequency equation. The $x_{nn}$ terms in the coefficient matrix are given as follows

$$x_{11} = \bar{\alpha}_p Z_n'(\bar{\alpha}_p a)$$
$$x_{12} = -\xi Z_n'(\bar{\beta}_p a)$$
$$x_{13} = \frac{n}{a} Z_n(\bar{\beta}_p a)$$
$$x_{14} = -\bar{\alpha}_p H_n^{(2)}(\bar{\alpha}_p a)$$
$$x_{15} = \xi H_n^{(2)}(\bar{\beta}_p a)$$
$$x_{16} = -\frac{n}{a} H_n^{(2)}(\bar{\beta}_p a)$$

(3-112)

$$x_{21} = -\frac{n}{a} Z_n(\bar{\alpha}_p a)$$
$$x_{22} = \frac{n}{a} Z_n(\bar{\beta}_p a)$$
$$x_{23} = -\bar{\beta}_p Z_n'(\bar{\beta}_p a)$$
$$x_{24} = \frac{n}{a} H_n^{(2)}(\bar{\alpha}_p a)$$
$$x_{25} = -\frac{n}{a} H_n^{(2)}(\bar{\beta}_p a)$$
$$x_{26} = \bar{\beta}_p H_n^{(2)}(\bar{\beta}_p a)$$

(3-113)
\[ x_{11} = -i \xi Z_n(\overline{\alpha}_\rho a) \]
\[ x_{12} = -i \overline{\beta}_\rho Z_n(\overline{\beta}_\rho a) \]
\[ x_{13} = 0 \]
\[ x_{14} = i \xi H_n^{(1)}(\overline{\alpha}_\rho a) \]
\[ x_{15} = i \overline{\beta}_\rho H_n^{(2)}(\overline{\beta}_\rho a) \]
\[ x_{16} = 0 \]  

(3-114)

\[ x_{41} = \left[ -\lambda (\overline{\alpha}_\rho^2 + \xi^2) + 2 \mu \left( \frac{n^2}{a^2} - \overline{\alpha}_\rho^2 \right) \right] Z_n(\overline{\alpha}_\rho a) - 2 \mu \frac{\overline{\alpha}_\rho}{a} Z_n'(\overline{\alpha}_\rho a) \]
\[ x_{42} = 2 \mu \xi \left[ \left( \overline{\beta}_\rho - \frac{n^2}{\overline{\beta}_\rho a^2} \right) Z_n(\overline{\beta}_\rho a) + \frac{1}{r} Z_n'(\overline{\beta}_\rho a) \right] \]
\[ x_{43} = 2 \mu \frac{n}{a} \left[ \overline{\beta}_\rho Z_n'(\overline{\beta}_\rho a) - \frac{1}{r} Z_n(\overline{\beta}_\rho a) \right] \]
\[ x_{44} = \left[ -\lambda (\overline{\alpha}_\rho^2 + \xi^2) + 2 \mu \left( \frac{n^2}{a^2} - \overline{\alpha}_\rho^2 \right) \right] H_n^{(1)}(\overline{\alpha}_\rho a) + 2 \mu \frac{\overline{\alpha}_\rho}{r} H_n^{(2)}(\overline{\alpha}_\rho a) \]
\[ x_{45} = -2 \mu \xi \left[ \left( \overline{\beta}_\rho - \frac{n^2}{\overline{\beta}_\rho a^2} \right) H_n^{(1)}(\overline{\beta}_\rho a) + \frac{1}{r} H_n^{(2)}(\overline{\beta}_\rho a) \right] \]
\[ x_{46} = -2 \mu \frac{n}{a} \left[ \overline{\beta}_\rho H_n^{(2)}(\overline{\beta}_\rho a) - \frac{1}{a} H_n^{(2)}(\overline{\beta}_\rho a) \right] \]  

(3-115)
\[\begin{align*}
  x_{51} &= \mu_p \frac{2n\xi}{a} \left[ \frac{1}{a} Z_n(\alpha_p, a) - \alpha_p Z'_n(\alpha_p, a) \right] \\
  x_{52} &= \mu_p \frac{2n\xi}{a} \left[ Z'_n(\beta_p, a) - \frac{1}{\beta_p^2} Z_n(\beta_p, a) \right] \\
  x_{53} &= \mu_p \left[ \frac{2\beta_p}{a} Z'(\beta_p, a) - \left( \frac{2n^2}{a^2} - \beta_p^2 \right) Z_n(\beta_p, a) \right] \\
  x_{54} &= -\mu_i \frac{2n\xi}{a} \left[ \frac{1}{a} H^{(1)}_n(\alpha, a) - \alpha H'^{(2)}_n(\alpha, a) \right] \\
  x_{55} &= -\mu_i \frac{2n\xi}{a} \left[ H'^{(2)}_n(\beta, a) - \frac{1}{\beta_i^2} H^{(2)}_n(\beta, a) \right] \\
  x_{56} &= -\mu_i \left[ \frac{2\beta_i}{a} H'^{(1)}_n(\beta, a) + \left( \frac{2n^2}{a^2} - \beta_i^2 \right) H^{(1)}_n(\beta, a) \right] \\
  x_{61} &= -i2\mu_p\xi\alpha_p Z'_n(\alpha_p, a) \\
  x_{62} &= i\mu_p \left( \frac{\xi^2}{a} - \beta_p^2 \right) Z'_n(\beta_p, a) \\
  x_{63} &= -i\mu_p \frac{n\xi}{a} Z_n(\beta_p, a) \\
  x_{64} &= i\mu_i 2\xi\alpha_i H'^{(1)}_n(\alpha, a) \\
  x_{65} &= -i\mu_i \left( \frac{\xi^2}{a^2} - \beta_i^2 \right) H^{(1)}_n(\beta, a) \\
  x_{66} &= i\mu_i \frac{n\xi}{a} H^{(1)}_n(\beta, a)
\end{align*}\]  

(3-117)

3.2.10 General Frequency Equation

The determinant of the coefficient matrix in Equation (3-111) is denoted by \( D_f \), and shown in Equation (3-118). The general frequency equation is obtained by setting \( D_f \) equal to zero, which results in a transcendental relationship between the
non-dimensional frequency and the non-dimensional wave number. The general
frequency equation encompasses longitudinal, torsional and flexural modes.

\[
\begin{bmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{bmatrix}
\begin{bmatrix}
  \nu \\
  \eta
\end{bmatrix}
= 0 \quad (3-118)
\]

3.3 DECOMPOSITION OF THE FREQUENCY EQUATION

3.3.1 Axisymmetric Motion

Axisymmetric motion refers to motion that is independent of the angular
coordinate. Axisymmetric modes are obtained by setting \( n \) equal to zero. When \( n = 0 \),
the general frequency equation given by the determinant in Equation (3-118) reduces to

\[
\begin{vmatrix}
  x_{11} & x_{12} & 0 & x_{14} & x_{15} & 0 \\
  0 & 0 & x_{33} & 0 & 0 & x_{36} \\
  x_{31} & x_{32} & 0 & x_{34} & x_{35} & 0 \\
  x_{41} & x_{42} & 0 & x_{44} & x_{45} & 0 \\
  0 & 0 & x_{53} & 0 & 0 & x_{56} \\
  x_{61} & x_{62} & 0 & x_{64} & x_{65} & 0
\end{vmatrix}
= 0 \quad (3-119)
\]
Interchanging rows and columns, the above determinant may be simplified into

$$\begin{vmatrix} x_{11} & x_{12} & x_{14} & x_{15} & 0 & 0 \\ x_{31} & x_{32} & x_{34} & x_{35} & 0 & 0 \\ x_{41} & x_{42} & x_{44} & x_{45} & 0 & 0 \\ x_{61} & x_{62} & x_{64} & x_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{23} & x_{26} \\ 0 & 0 & 0 & 0 & x_{33} & x_{36} \end{vmatrix} = 0 \quad (3-120)$$

The determinant can then be decomposed into two sub-determinants given by

$$\begin{vmatrix} x_{11} & x_{12} & x_{14} & x_{15} \\ x_{31} & x_{32} & x_{34} & x_{35} \\ x_{41} & x_{42} & x_{44} & x_{45} \\ x_{61} & x_{62} & x_{64} & x_{65} \end{vmatrix} \times \begin{vmatrix} x_{23} & x_{26} \\ x_{33} & x_{36} \end{vmatrix} = 0 \quad (3-121)$$

The sub-determinant, $D_2$, represents the frequency equation for longitudinal modes. Only displacement components $u_r$ and $u_z$ are non-zero, and the motion involves coupled dilatational and equivoluminal displacements (McNiven, Sackman and Shah, 1963). The sub-determinant, $D_1$, gives purely torsional modes with $u_r = u_z = 0$. The frequencies of the longitudinal modes are generally dependent on the Poisson’s ratios of the pile and soil, but the frequencies of the torsional modes are independent of the Poisson’s ratios (Armenakas, 1967).
3.3.2 Motion Independent of the Axial Coordinate

When the wave number, \( \xi = 0 \), the exponential terms in the displacements given by Equations (3-64) to (3-70) do not depend on the axial coordinate, \( z \), and therefore, the motion becomes independent of \( z \). The wavelength, \( \lambda = 2\pi/\xi \), becomes infinite and all points in the pile move in-phase with respect to the \( z \)-axis. Substituting \( \xi = 0 \) into the general frequency equation given by the determinant in Equation (3-118) gives

\[
\begin{vmatrix}
    x_{11} & 0 & x_{13} & x_{14} & 0 & x_{16} \\
    x_{21} & 0 & x_{23} & x_{24} & 0 & x_{26} \\
    0 & x_{32} & 0 & 0 & x_{35} & 0 \\
    x_{41} & 0 & x_{43} & x_{44} & 0 & x_{46} \\
    x_{51} & 0 & x_{53} & x_{54} & 0 & x_{56} \\
    0 & x_{62} & 0 & 0 & x_{65} & 0 \\
\end{vmatrix} = 0 \quad (3-122)
\]

Interchanging rows and columns, the above determinant may be simplified into

\[
\begin{vmatrix}
    x_{11} & x_{13} & x_{14} & x_{16} & 0 & 0 \\
    x_{21} & x_{23} & x_{24} & x_{26} & 0 & 0 \\
    x_{41} & x_{43} & x_{44} & x_{46} & 0 & 0 \\
    x_{51} & x_{53} & x_{54} & x_{56} & 0 & 0 \\
    0 & 0 & 0 & 0 & x_{32} & x_{35} \\
    0 & 0 & 0 & 0 & x_{62} & x_{65} \\
\end{vmatrix} = 0 \quad (3-123)
\]
The determinant can then be decomposed into two sub-determinants given by

\[
\begin{vmatrix}
    x_{11} & x_{13} & x_{14} & x_{16} \\
    x_{21} & x_{23} & x_{24} & x_{26} \\
    x_{31} & x_{33} & x_{34} & x_{36} \\
    x_{51} & x_{53} & x_{54} & x_{56}
\end{vmatrix}_{\delta_i} \times
\begin{vmatrix}
    x_{32} & x_{35} \\
    x_{62} & x_{65}
\end{vmatrix}_{\eta_i} = 0 \tag{3-124}
\]

The sub-determinant, \( D_j \), corresponds to the frequency equation for plane strain motion involving displacements components, \( u_r \) and \( u_\theta \) but with \( u_z = 0 \). The displacements are associated with both dilatational and equivoluminal components. The sub-determinant, \( D_\delta \), corresponds to longitudinal shear motion associated only with the \( u_z \) component. The frequency of these solutions are independent of the Poisson’s ratios (Armenakas, 1967).

### 3.3.3 Axisymmetric Motion Independent of the Axial Coordinate

The motion becomes independent of the angular coordinate, \( \theta \), and the axial coordinate, \( z \), when \( n = 0 \) and \( \xi = \theta \), respectively. Substituting these into the general frequency equation given by the determinant in Equation (3-118) gives
\[
\begin{vmatrix}
 x_{11} & 0 & 0 & x_{14} & 0 & 0 \\
 0 & 0 & x_{23} & 0 & 0 & x_{26} \\
 0 & x_{32} & 0 & 0 & x_{35} & 0 \\
 x_{41} & 0 & 0 & x_{44} & 0 & 0 \\
 0 & 0 & x_{53} & 0 & 0 & x_{56} \\
 0 & x_{62} & 0 & 0 & x_{65} & 0 \\
\end{vmatrix}
= 0 \quad (3-125)
\]

Interchanging rows and columns, the above determinant may be simplified into

\[
\begin{vmatrix}
 x_{11} & x_{14} & 0 & 0 & 0 & 0 \\
 x_{41} & x_{44} & 0 & 0 & 0 & 0 \\
 0 & 0 & x_{23} & x_{26} & 0 & 0 \\
 0 & 0 & x_{33} & x_{56} & 0 & 0 \\
 0 & 0 & 0 & 0 & x_{32} & x_{35} \\
 0 & 0 & 0 & 0 & x_{62} & x_{65} \\
\end{vmatrix}
= 0 \quad (3-126)
\]

The determinant can then be decomposed into three sub-determinants given by

\[
\begin{vmatrix}
 x_{32} & x_{35} \\
 x_{62} & x_{65} \\
\end{vmatrix}
\begin{vmatrix}
 x_{11} & x_{14} \\
 x_{41} & x_{44} \\
\end{vmatrix}
\begin{vmatrix}
 x_{23} & x_{26} \\
 x_{33} & x_{56} \\
\end{vmatrix}
= 0 \quad (3-127)
\]

The sub-determinant, \( D_3 \), represents the frequency equation corresponding to longitudinal shear motion involving only the \( u_z \) component. The dilatational and rotational motions are uncoupled for plane strain axisymmetric waves. The frequency
equation for plane strain extensional waves, which consists only of radial
displacements, is given by the sub-determinant, $D_\alpha$. The sub-determinant, $D_\alpha$,
describes the frequency equation for plane strain shear motion, containing only $u_\alpha$
displacements (Armenakas, 1967).

3.4 SUMMARY

The general frequency equation for an infinitely long cylindrical pile embedded
in soil was developed from the basic equations of elasticity. Appropriate potential
functions were selected to account for the assumed harmonic wave propagation along
the pile and outward, decaying wave propagation in the soil. The general expressions
for displacements and stresses in the pile and soil were derived. The lateral boundary
conditions require the displacements and stresses at the pile-soil interface to be
continuous. Application of the boundary conditions resulted in a system of
homogeneous equations with unknown constants. The homogeneous system of
equations has a non-trivial solution only if the determinant vanishes. The general
frequency equation for an embedded pile was obtained by setting the determinant to
zero.

Taking into account various motions in the pile, the general frequency equation
decomposed into products of simpler sub-determinants. The frequency equations for
longitudinal and torsional modes were obtained by considering axisymmetric motion.
When motion independent of the axial coordinate was investigated, the frequency equations for plane strain motion and longitudinal shear motion were attained. Examining axisymmetric motion independent of the axial coordinate yielded the frequency equations for longitudinal shear motion, plane strain extensional motion and plane strain shear motions.
4.1 INTRODUCTION

This chapter deals specifically with longitudinal modes in a cylindrical pile embedded in soil. The frequency equation for longitudinal modes developed in Chapter 3 is reworked and expressed explicitly in a non-dimensional form. The non-dimensional variables in the frequency equation can be defined in terms of the Poisson’s ratios, shear moduli and densities of the pile and soil. The frequency equation thus constitutes a transcendental relationship between the non-dimensional frequency, $\Omega$ and non-dimensional wave number, $\xi_a$, for a given set of geometric and physical properties of the pile and soil.

The equations for axial and radial displacements in the pile and soil are derived. The equations for radial and shear stresses in the pile are also determined. The constant coefficients in the equations for displacements and stresses are not unique and have to be determined separately for each guided wave mode. A normalizing factor, which sets the energy flux or power of any mode through a cross-section of the pile to be unity, is derived so that displacements or stresses of any mode can be
compared directly. The phase and group velocities of guided wave modes are defined and the determination of these velocities from the dispersion curves is shown.

4.2 FREQUENCY EQUATION FOR LONGITUDINAL MODES

Referring to Section 3.2.9, the linear system of equations governing longitudinal axisymmetric motion in the pile, can be written in matrix notation as

\[
\begin{bmatrix}
x_{11} & x_{12} & x_{14} & x_{15} \\
x_{31} & x_{32} & x_{34} & x_{35} \\
x_{41} & x_{42} & x_{44} & x_{45} \\
x_{61} & x_{62} & x_{64} & x_{65}
\end{bmatrix}
\begin{bmatrix}
A_{l_1} \\
A_{l_2} \\
A_{l_3} \\
A_{l_4}
\end{bmatrix}
= 0
\]  \hspace{1cm} (4-1)

where the first two and last two rows represent the displacement and stress boundary conditions, respectively. The determinant, \( D \), in Equation (3-121) corresponds to the determinant of the coefficient matrix shown in Equation (4-1). This determinant represents the frequency equation for longitudinal modes and is given by

\[
\begin{vmatrix}
x_{11} & x_{12} & x_{14} & x_{15} \\
x_{31} & x_{32} & x_{34} & x_{35} \\
x_{41} & x_{42} & x_{44} & x_{45} \\
x_{61} & x_{62} & x_{64} & x_{65}
\end{vmatrix}
= 0
\]  \hspace{1cm} (4-2)

By substituting the \( x_{mn} \) terms from Equations (3-113) to (3-118) into Equation (4-2), the expanded determinant is obtained and shown in Equation (4-3).
\[
\begin{align*}
\alpha, \bar{Z}^0(\bar{\alpha}, a) & \quad -\xi \bar{Z}(\bar{\alpha}, a) & -\bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & \xi H^{(1)}_2(\bar{\alpha}, a) \\
-i\xi Z_0(\bar{\alpha}, a) & -i\bar{\alpha} H_{\alpha}^{(1)}(\bar{\alpha}, a) & i\bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & i\bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) \\
-\left[\lambda_c (\bar{\alpha}^2 + \xi^2) + 2\mu_c \bar{\alpha}^2\right] Z_0(\bar{\alpha}, a) & 2\mu_c \left[\bar{\alpha}_c Z_0(\bar{\alpha}, a) + \frac{3}{a} Z_0(\bar{\alpha}, a)\right] & \left[\lambda_c (\bar{\alpha}^2 + \xi^2) + 2\mu_c \bar{\alpha}^2\right] H^{(1)}_0(\bar{\alpha}, a) & -2\mu_c \left[\bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) + \frac{3}{a} H^{(1)}_0(\bar{\alpha}, a)\right] \\
-2\mu_c \bar{\alpha} Z_0(\bar{\alpha}, a) & -2\mu_c \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & -2\mu_c \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & -2\mu_c \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) \\
-i2\mu_c \xi \bar{\alpha} Z_0(\bar{\alpha}, a) & i2\mu_c \xi \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & i2\mu_c \xi \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a) & -i2\mu_c \xi \bar{\alpha} H^{(1)}_0(\bar{\alpha}, a)
\end{align*}
\]

\[= 0 \quad (4-3)\]
\[
\begin{align*}
\bar{\alpha}, \text{a} Z_2^a(\bar{\alpha}, a) & \quad - \xi a Z_2^a(\bar{\beta}, a) & \quad - \bar{\alpha}, a H_s^{(3)}(\bar{\alpha}, a) & \quad \xi a H_s^{(3)}(\bar{\beta}, a) \\
- \xi a Z_2^a(\bar{\alpha}, a) & \quad - \bar{\beta}, a Z_2^a(\bar{\beta}, a) & \quad \xi a H_s^{(3)}(\bar{\alpha}, a) & \quad \bar{\beta}, a H_s^{(3)}(\bar{\beta}, a) \\
- \left[ \frac{\lambda_s}{\mu_p} (\bar{\alpha}, a^2 + x^2 a^2) + 2 \bar{a} a^2 \right] Z_2^a(\bar{\alpha}, a) & \quad 2 \xi a [\bar{\beta}, a Z_2^a(\bar{\beta}, a)] + Z_2^a(\bar{\beta}, a) & \quad \left[ \frac{\lambda_s}{\mu_p} (\bar{\alpha}, a^2 + x^2 a^2) + 2 \frac{\mu_a}{\mu_p} \bar{\alpha}, a^2 \right] H_s^{(3)}(\bar{\alpha}, a) & \quad - 2 \frac{\mu_a}{\mu_p} \xi a [\bar{\beta}, a H_s^{(3)}(\bar{\beta}, a)] + H_s^{(3)}(\bar{\beta}, a) \\
- 2 \xi a \bar{\alpha}, a Z_2^a(\bar{\alpha}, a) & \quad [\xi^2 a^2 - \bar{\beta}, a^2] Z_2^a(\bar{\beta}, a) & \quad 2 \frac{\mu_a}{\mu_p} \xi a \bar{\alpha}, a H_s^{(3)}(\bar{\alpha}, a) & \quad - \frac{\mu_a}{\mu_p} \left[ \xi^2 a^2 - \bar{\beta}, a^2 \right] H_s^{(3)}(\bar{\beta}, a)
\end{align*}
\]

= 0 \quad (4-4)
This determinant is rendered dimensionless by multiplying Row 1 by \( a \), Row 2 by \( a/i \), Row 3 by \( a^2/\mu_p \) and Row 4 by \( a^2/i\mu_p \). The resulting non-dimensional determinant is shown in Equation (4-4). The following substitutions are introduced following the notation in Thurston (1978),

\[
Y = \xi a \quad (4-5)
\]
\[
V = \overline{\alpha}_p a \quad (4-6)
\]
\[
W = \overline{\alpha}_a a \quad (4-7)
\]
\[
X = \overline{\beta}_p a \quad (4-8)
\]
\[
U = \overline{\beta}_a a \quad (4-9)
\]

where \( Y \) is the non-dimensional wave number; and \( V, W, X \) and \( U \) are non-dimensional terms. Substituting Equations (4-5) to (4-9) into Equation (4-4) enables the dimensionless determinant to be expressed in a simplified form, as shown in Equation (4-10). The determinant is further reduced by replacing the special functions \( Z'_0 \) and \( H^{(2)}_0 \) by \( -Z_l \) and \( -H^{(2)}_l \) respectively. The final form of the determinant representing the frequency equation for longitudinal modes is shown in Equation (4-11).
\[
\begin{array}{cccc}
& vZ'_x(v') & -vZ'_y(x) & -WH'_0^{(0)}(w') & YH'_0^{(0)}(u) \\
& -vZ'_y(v') & -xZ'_x(x) & YH'_0^{(0)}(w) & UH'_0^{(0)}(u) \\
& -2v^2Z'_x(v') & \left[2v^2 - x^2\right]Z'_x(x) & 2wH'_0^{(0)}(w) & -\frac{\mu_s}{\mu_p} [UH'_0^{(0)}(U) + H'_0^{(0)}(U)] \\
& -2v^2Z'_y(v') & \left[2v^2 - x^2\right]Z'_y(x) & 2\frac{\mu_s}{\mu_p} [wH'_0^{(0)}(w) + \frac{\mu_s}{\mu_p} UH'_0^{(0)}(U) - \frac{\mu_s}{\mu_p} UH'_0^{(0)}(U)] & -\frac{\mu_s}{\mu_p} [v^2 - U^2]H'_0^{(0)}(U)
\end{array}
\]

\[
\begin{array}{cccc}
& -vZ'_x(v') & vZ'_y(x) & WH'_0^{(0)}(w) & -YH'_0^{(0)}(u) \\
& -vZ'_y(v') & -xZ'_x(x) & YH'_0^{(0)}(w) & UH'_0^{(0)}(u) \\
& -2v^2Z'_x(v') & \left[2v^2 - x^2\right]Z'_x(x) & 2wH'_0^{(0)}(w) & -\frac{\mu_s}{\mu_p} [UH'_0^{(0)}(U) - H'_0^{(0)}(U)] \\
& 2v^2Z'_y(v') & -\left[2v^2 - x^2\right]Z'_y(x) & -\frac{\mu_s}{\mu_p} [wH'_0^{(0)}(w) + \frac{\mu_s}{\mu_p} UH'_0^{(0)}(U) - \frac{\mu_s}{\mu_p} UH'_0^{(0)}(U)] & \frac{\mu_s}{\mu_p} [v^2 - U^2]H'_0^{(0)}(U)
\end{array}
\]
4.3 FREQUENCY EQUATION VARIABLES

The reduction of the frequency equation into the simplified dimensionless form shown in Equation (4-11) requires the definition of the substitutions used in Equations (4-5) to (4-9) so that the equation can be solved numerically. Equations (3-23) and (3-32) from Chapter 3 are multiplied by $a^2$ and reintroduced as

$$\left( \bar{\alpha}_p a \right)^2 = \frac{(\omega a)^2}{c_{l_p}^2} - (\xi a)^2 \tag{4-12}$$

$$\left( \bar{\alpha}_q a \right)^2 = \frac{(\omega a)^2}{c_{l_q}^2} - (\xi a)^2 \tag{4-13}$$

$$\left( \bar{\beta}_p a \right)^2 = \frac{(\omega a)^2}{c_{r_p}^2} - (\xi a)^2 \tag{4-14}$$

$$\left( \bar{\beta}_q a \right)^2 = \frac{(\omega a)^2}{c_{r_q}^2} - (\xi a)^2 \tag{4-15}$$

The non-dimensional frequency, $\Omega$, is related to the angular frequency, $\omega$, in the following manner

$$\Omega = \frac{\omega a}{c_{r_r}} \tag{4-16}$$
Substituting Equations (4-5) to (4-9) and (4-16) into Equations (4-12) to (4-15) gives

\[ V^2 = \frac{c_{l_p}^2}{c_{l_p}} \Omega^2 - Y^2 \]  \hspace{1cm} (4-17)

\[ X^2 = \Omega^2 - Y^2 \]  \hspace{1cm} (4-18)

\[ W^2 = \frac{c_{l_r}^2}{c_{l_r}} \Omega^2 - Y^2 \]  \hspace{1cm} (4-19)

\[ U^2 = \frac{c_{r_p}^2}{c_{r_p}} \Omega^2 - Y^2 \]  \hspace{1cm} (4-20)

The ratios of the bulk wave velocities in the pile and soil are denoted by \( \kappa_p \) and \( \kappa_s \), respectively. These ratios can be expressed in terms of the Poisson’s ratio and are given by

\[ \kappa_p = \frac{c_{r_p}}{c_{l_p}} \]

\[ = \left[ \frac{1 - 2\nu_p}{2(1 - \nu_p)} \right]^\frac{1}{2} \]  \hspace{1cm} (4-21)
\[ \kappa_s = \frac{c_{r_p}}{c_{r_s}} \]

\[ = \left[ \frac{1 - 2\nu_s}{2(1 - \nu_s)} \right]^{\frac{1}{2}} \] (4-22)

where \( \nu_p \) and \( \nu_s \) are the Poisson's ratios for the pile and soil, respectively. The ratio of the shear wave velocities in the pile and soil, \( \eta \), is given by

\[ \eta = \frac{c_{r_p}}{c_{r_s}} \] (4-23)

Substituting Equations (4-21), (4-22) and (4-23) into Equations (4-17) to (4-20) gives

\[ \nu^2 = \kappa_s^2 \Omega^2 - Y^2 \] (4-24)

\[ X^2 = \Omega^2 - Y^2 \] (4-25)

\[ W^2 = \eta^2 \kappa_s^2 \Omega^2 - Y^2 \] (4-26)

\[ U^2 = \eta^2 \Omega^2 - Y^2 \] (4-27)
Thus, the non-dimensional terms, \( V \), \( W \), \( X \) and \( U \) are obtained by specifying the variables \( \eta \), \( \kappa_p \) and \( \kappa_s \). The unknowns, \( \Omega \) and \( \gamma \) are the non-dimensional frequency and wave number, which are determined numerically. The shear wave velocity ratio can be expressed as

\[
\eta^2 = \frac{c_{l_s}^2}{c_{r_s}^2} = \frac{\mu_p}{\rho_p} \left/ \frac{\mu_s}{\rho_s} \right. \\
= \frac{\mu_p}{\mu_s} \times \frac{\rho_s}{\rho_p}
\]  

(4-28)

where \( \mu_p \) and \( \mu_s \) are the shear moduli of the pile and soil, respectively, and \( \rho_p \) and \( \rho_s \) are the densities of the pile and soil, respectively. The bulk wave velocities are given as

\[
c_{l_s} = \sqrt{\frac{\lambda_s + 2\mu_s}{\rho_s}}
\]  

(4-29)

\[
c_{r_s} = \sqrt{\frac{\mu_s}{\rho_s}}
\]  

(4-30)

where \( \lambda_s \) and \( \mu_s \) are the Lamé constants and \( \rho_s \) is the density.
Dividing Equation (4-29) by (4-30) and rearranging gives

\[
\frac{\lambda_i}{\mu_i} = \left(\frac{c_t}{c_r}\right)^2 - 2
\]

\[
= \frac{l}{\kappa_i^2} - 2
\]

(4-31)

The ratio of the Lamé constants for the pile appears in the frequency equation and it can be expressed as

\[
\frac{\lambda_p}{\mu_p} = \frac{l}{\kappa_p^2} - 2
\]

(4-32)

The ratio, $\frac{\lambda_i}{\mu_i}$, appears in the (3,3) component of the determinant in Equation (4-11) $\mu_p$ and it can be expressed in the following form

\[
\frac{\lambda_i}{\mu_p} = \frac{\lambda_i}{\mu_i} \times \frac{\mu_i}{\mu_p}
\]

\[
= \left(\frac{l}{\kappa_i^2} - 2\right) \times \frac{\mu_i}{\mu_p}
\]

(4-33)
The preceding section shows that the all the terms in the determinant, except the frequency and the wave number, can be determined if the pile properties, \( \mu_p \cdot \rho_p \) and \( \nu_p \), and the soil properties, \( \mu_s \), \( \rho_s \), and \( \nu_s \), are defined. Given these properties, the frequency equation reduces to a 4x4 determinant with only two unknowns - the non-dimensional frequency and non-dimensional wave number. The numerical solution to the frequency equation is described in Chapter 5.

4.4 DISPLACEMENTS

The displacements in the pile and soil for the general case are given by Equations (3-64) to (3-70). Considering the specific case of longitudinal modes, substituting \( n = 0 \) results in the angular displacement vanishing and leaving only the axial and radial displacements. Some of the terms in the remaining equations also vanish. The equations may be simplified further by replacing the special functions \( Z_n' \) and \( H_n^{(2)} \) by \( -Z_l \) and \( -H_l^{(2)} \), respectively.

The axial and radial displacement expressions contain the exponential term, \( e^{i(\omega t - \xi z)} \), which represents a harmonic wave propagating with a phase velocity, \( c = \omega / \xi \), in the positive \( z \)-direction. The exponential term, which is independent of the radial coordinate, does not affect the radial distribution of the axial and radial
displacements and hence, it may be ignored. The radial distribution of the axial and radial displacements in the pile and soil are given as

\[ u_z = -i \left[ A_{\alpha_p} \xi Z_0(\alpha_p r) + A_{\beta_p} \beta_p Z_0(\beta_p r) \right] \]  \hfill (4-34)

\[ u_r = \left[ -A_{\alpha_p} \bar{\alpha}_p Z_1(\alpha_p r) + A_{\beta_p} \xi Z_1(\beta_p r) \right] \]  \hfill (4-35)

\[ u_z = -i \left[ A_{\alpha_p} \xi H_0^{(2)}(\alpha_p r) + A_{\beta_p} \beta_p H_0^{(2)}(\beta_p r) \right] \]  \hfill (4-36)

\[ u_r = \left[ -A_{\alpha_p} \bar{\alpha}_p H_1^{(2)}(\alpha_p r) + A_{\beta_p} \xi H_1^{(2)}(\beta_p r) \right] \]  \hfill (4-37)

The radius, \( r \), can be normalized by introducing the following substitution

\[ \frac{r}{a} = R \]  \hfill (4-38)

where \( R \) is the non-dimensional radius and \( a \) is the radius of the pile. Thus, the range \( 0 \leq r \leq a \) may be replaced by \( 0 \leq R \leq 1 \). It may be noted from Equations (4-6) to (4-9) that \( \overline{\alpha}_p r, \overline{\alpha}_r, \overline{\beta}_p r \) and \( \overline{\beta}_r \) will be replaced by \( VR, WR, XR \) and \( UR \), respectively.
The axial and radial displacements in the pile and soil are expressed in non-dimensional terms by multiplying Equations (4-35) to (4-36) by \( a \) and substituting Equations (4-5) to (4-9) and Equation (4-38) into the resulting expressions. Rearranging the final expressions, the normalized displacements, \( \bar{u} \), are given by

\[
\bar{u}_{z_p} = -\frac{i}{a} \left[ A_{r_p} Y Z_o(VR) + A_{r_j} X Z_o(XR) \right]
\]  
(4-39)

\[
\bar{u}_r = \frac{l}{a} \left[ -A_{r_p} Y Z_i(VR) + A_{r_j} Y Z_i(XR) \right]
\]  
(4-40)

\[
\bar{u}_z = -\frac{i}{a} \left[ A_{r} Y H^{(2)}(WR) + A_{r_j} U H^{(2)}(UR) \right]
\]  
(4-41)

\[
\bar{u}_\tau = \frac{l}{a} \left[ -A_{r} W H_i^{(2)}(WR) + A_{r_j} Y H_i^{(2)}(UR) \right]
\]  
(4-42)

4.5 **STRESSES**

The stresses in the pile for the general case are given by Equations (3-104) to (3-106). As in the case of displacements, substituting \( n = 0 \) results in \( r_{rr_p} \) vanishing and leaving only \( \sigma_{rr_p} \) and \( r_{rr_p} \). The equations are further simplified by replacing the special functions \( Z_o' \) and \( H_o^{(2)} \) by \(-Z_i\) and \(-H_i^{(2)}\), respectively, and ignoring the
exponential term as explained in Section 4.4. The stresses in the pile are then given as

\[
\sigma_{\tau_{_p}} = A_{_f} \left\{ -\lambda_p \left( \alpha^2_p + \xi^2 \right) - 2\mu_p \alpha^2_p \right\} Z_0(\alpha_p r) + 2\mu_p \frac{\alpha_p}{r} Z_1(\alpha_p r) \\
+ A_{_f} \frac{2\mu_p}{r} \left[ \beta_p Z_0(\beta_p r) - \frac{r}{\beta_p} Z_1(\beta_p r) \right]
\] (4-43)

\[
\tau_{\tau_{_p}} = i\mu_p \left\{ A_{_f} \left[ 2\xi \alpha_p Z_1(\alpha_p r) \right] - A_{_f} \left[ \xi^2 \right] Z_1(\beta_p r) \right\}
\] (4-44)

Similarly, multiply Equations (4-43) and (4-44) by \( \alpha^2 \) and substitute Equations (4-6) to (4-9) and Equation (4-38) into the resulting expressions. Rearranging the final expressions give the normalized stresses. \( \bar{\sigma} \) and \( \bar{\tau} \), in the pile, expressed in non-dimensional terms as

\[
\bar{\sigma}_{\tau_{_p}} = \frac{\mu_p}{\alpha^2} \left\{ A_{_f} \left\{ -\frac{\lambda_p}{\mu_p} \left( V^2 + Y^2 \right) + 2V^2 \right\} Z_0(VR) + \frac{2V}{R} Z_1(VR) \\
+ A_{_f} \cdot 2Y \left[ XZ_0(XR) - \frac{r}{R} Z_1(XR) \right] \right\}
\] (4-45)

\[
\bar{\tau}_{\tau_{_p}} = \frac{i\mu_p}{\alpha^2} \left\{ A_{_f} \left[ 2YVZ_1(VR) \right] - A_{_f} \left[ \left( Y^2 - X^2 \right) Z_1(XR) \right] \right\}
\] (4-46)
4.6 ENERGY

The radial distributions of the axial and radial displacements of any mode can be evaluated from the expressions given in Equations (4-39) to (4-42). However, the unknown constants in these equations, \( A_{r} \), \( A_{s} \), \( A_{l} \), and \( A_{t} \), are to be determined separately for each mode. The linear system of equations governing longitudinal axisymmetric motion in the pile, can be written in matrix notation as

\[
\begin{bmatrix}
    x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\
    x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\
    x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\
    x_{41} & x_{42} & x_{43} & x_{44} & x_{45} \\
    x_{51} & x_{52} & x_{53} & x_{54} & x_{55}
\end{bmatrix}
\begin{bmatrix}
    A_{l} \\
    A_{r} \\
    A_{s} \\
    A_{l} \\
    A_{t}
\end{bmatrix}
= 0
\]

(4-47)

Because the system of equations is homogeneous and the determinant is equated to zero to get the non-trivial solutions, the unknown constants are obtained by solving for the null space vector. The solutions obtained, however, are not the absolute values. Therefore, it is not possible to compare the displacements from different modes directly. This problem can be overcome if the energy of the modes were uniform. Thus, by normalizing the energy in the modes to unity and applying the normalization to the displacements, a direct comparison of the displacements from different modes can be made. The following section shows the derivation of energy in a guided wave mode and the determination of the normalizing constant.
Neglecting body forces, the energy flux or power entering or leaving the embedded pile over a surface, \( S \) is given by,

\[
P = \oint_S t_k \dot{u}_k \, dA = \oint_S \sigma_{ik} n_i \dot{u}_k \, dA = \oint_S (\sigma_{ik} \dot{u}_k) n_i \, dA
\]  

(4-48)

where \( p \) is the power, and, in indicial notation, \( t_k \) is the traction, \( \dot{u}_k \) is the particle velocity, \( \sigma_{ik} \) is the stress tensor and \( n_i \) is the unit normal vector to the surface. The average rate of flow of energy, or average power, in a plane time–harmonic wave may be calculated by forming the vector product of the traction and the particle velocity and taking the time average of this product (Achenbach, 1973). The average power, \( \langle p \rangle \), is given by

\[
\langle p \rangle = \left\langle \oint_S \sigma_{ik} \dot{u}_k n_i \, dA \right\rangle
\]  

(4-49)

\[
= \oint_S \langle \sigma_{ik} \dot{u}_k \rangle n_i \, dA
\]

where \( \langle \ldots \rangle \) refers to the time average over a period, and,

\[
\langle \sigma_{ik} \dot{u}_k \rangle = \frac{1}{T} \int_0^T \sigma_{ik} \dot{u}_k \, dt
\]  

(4-50)

where \( t \) is the time at any given instant and \( T \) is the period.
The expressions for displacements and stresses in the pile are given in Sections 3.2.5 and 3.2.7. The displacement, particle velocity and stress components can also be expressed in general as

\[ u_k = u_k(r) e^{i(\omega t - \xi z)} \]  
\[ \dot{u}_k = \dot{u}_k(r) e^{i(\omega t - \xi z)} \]  
\[ = i\omega u_k(r) e^{i(\omega t - \xi z)} \]  
\[ \sigma_{ik} = \sigma_{ik}(r) e^{i(\omega t - \xi z)} \]  

where \( u_k, \dot{u}_k \) and \( \sigma_{ik} \) are the displacement, velocity and stress components, respectively, and \( u_k(r), \dot{u}_k(r) \) and \( \sigma_{ik}(r) \) are functions of the radius. The velocity is obtained by taking the time derivative of the displacement.

In the above expressions, \( \omega \) is real while \( u_k(r), \sigma_{ik}(r) \) and \( \xi \) are complex. Thus when computing power, in the physical sense, only the real part of the stress and velocity are considered in the computations. The real part of a complex variable is given by

\[ \text{Re}[z] = \frac{1}{2} \left[ z + z^* \right] \]
where \( z \) is complex and \( z^\ast \) is the complex conjugate of \( z \). Evaluating the product of the real part of stress and the real part of particle velocity gives

\[
\text{Re}[\sigma_k] \times \text{Re}[\dot{u}_k] = \frac{1}{2} \left[ \sigma_k(r) e^{i(\omega t - \xi z)} + \sigma^\ast_k(r) e^{-i(\omega t - \xi^\ast z)} \right] \times \frac{1}{2} \left[ \dot{u}_k(r) e^{i(\omega t - \xi^\ast z)} + \dot{u}_k^\ast(r) e^{-i(\omega t - \xi z)} \right] (4-55)
\]

\[
= \frac{1}{4} \left[ \sigma_k(r) \dot{u}_k(r) e^{i2(\omega t - \xi z)} + \sigma_k^\ast(r) \dot{u}_k^\ast(r) e^{-i(\xi - \xi^\ast) z} \right] + \sigma^\ast_k(r) \dot{u}_k^\ast(r) e^{-i2(\omega t - \xi^\ast z)} \right] (4-55)
\]

The average power is then obtained by substituting Equations (4-55) into (4-50) which gives

\[
\langle \sigma_k \dot{u}_k \rangle = \frac{1}{T} \int_0^T \frac{1}{4} \left[ \sigma_k(r) \dot{u}_k(r) e^{i2(\omega t - \xi z)} + \sigma_k^\ast(r) \dot{u}_k^\ast(r) e^{-i(\xi - \xi^\ast) z} \right] dt (4-56)
\]

The terms in Equation (4-56) that contain the time harmonic variable \( e^{i\omega t} \) vanish when integrated over a period giving

\[
\langle \sigma_k \dot{u}_k \rangle = \frac{1}{4} \left[ \sigma_k(r) \dot{u}_k^\ast(r) e^{-i(\xi - \xi^\ast) z} + \sigma_k^\ast(r) \dot{u}_k(r) e^{-i(\xi - \xi^\ast) z} \right] (4-57)
\]
The complex wave number can be represented as

$$\xi = \xi_r + i\xi_i$$  \hspace{1cm} (4-58)

where $\xi_r$ and $\xi_i$ are the real and imaginary parts of the wave number, respectively.

Substituting Equation (4-58) into (4-57) gives

$$\langle \sigma_{\alpha} u_k \rangle = \frac{e^{z^2z'}}{4} \left[ \sigma_{\alpha} (r) u_k^* (r) + \sigma_{\alpha}^* (r) \dot{u}_k (r) \right]$$  \hspace{1cm} (4-59)

The average power can be expressed in terms of stresses and displacements by substituting Equation (4-52) into (4-59) which gives

$$\langle \sigma_{\alpha} u_k \rangle = \frac{1}{4} e^{z^2z'} \left[ -i\omega \sigma_{\alpha} (r) u_k^* (r) + i\omega \sigma_{\alpha}^* (r) u_k (r) \right]$$

$$= \frac{\omega}{4i} e^{z^2z'} \left[ \sigma_{\alpha} (r) u_k^* (r) - \sigma_{\alpha}^* (r) u_k (r) \right]$$  \hspace{1cm} (4-60)

Equation (4-49) gives the average power in the pile in indicial notation, which when expanded, yields nine components. In the case of longitudinal modes, the components containing $\theta$ vanish, leaving only two components in the axial direction and two components in the radial direction. The average power through the cross-
section of the pile, \( \langle P_z \rangle \) is thus given by

\[
\langle P_z \rangle = \oint_{S_z} \left[ \langle \tau_{zz} \rangle n_z + \langle \sigma_{zz} \rangle n_z \right] dA
\]  
(4-61)

where \( S_z \) is the surface area of the end of the pile. The average power through the sides of the pile, \( \langle P_r \rangle \) is given by

\[
\langle P_r \rangle = \oint_{S_r} \left[ \langle \sigma_{rr} \rangle n_r + \langle \tau_{rr} \rangle n_r \right] dA \bigg|_{r=a}
\]  
(4-62)

where \( S_r \) is the surface area of the curved side of the pile.

Consider a small slice of the embedded pile as shown in Figure 4-1. The power entering the pile from the top cross-section is balanced by the power leaving the pile through the bottom cross-section and through the sides. Applying this energy balance to Equations (4-61) and (4-62) gives

\[
\text{Power In} = \text{Power Out} \\
\left. \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} \langle \sigma_{kk} \hat{u}_k \rangle \hat{n}_i \, r \, dr \, d\theta \right|_z = \left. \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=a} \langle \sigma_{kk} \hat{u}_k \rangle \hat{n}_i \, r \, dr \, d\theta \right|_{z=\Delta z} + \left. \int_{\theta=0}^{\theta=\pi} \langle \sigma_{kk} \hat{u}_k \rangle \hat{n}_i \, \Delta z \, r \, d\theta \right|_{r=a}
\]  
(4-63)
Figure 4-1  Power Balance through a Small Slice in a Pile

Integrating Equation (4-63) with respect to $\theta$ gives

$$2\pi \int_{r=a}^{r=b} (\sigma_{ik} \dot{u}_k) \hat{n}_l \, r \, dr \bigg|_{z} = 2\pi \int_{r=a}^{r=b} (\sigma_{ik} \dot{u}_k) \hat{n}_l \, r \, dr \bigg|_{z+\Delta z} + 2m a (\sigma_{ik} \dot{u}_k) \hat{n}_l \, \Delta z \bigg|_{r=a} \quad (4-64)$$

Rearranging Equation (4-64) and dividing by $\Delta z$ gives

$$\int_{r=a}^{r=b} \frac{(\sigma_{ik} \dot{u}_k) \hat{n}_l}{\Delta z} \bigg|_{z+\Delta z} - (\sigma_{ik} \dot{u}_k) \hat{n}_l \bigg|_{z} \, r \, dr = -a (\sigma_{ik} \dot{u}_k) \hat{n}_l \bigg|_{r=a} \quad (4-65)$$
Taking the limit as $\Delta z \rightarrow 0$, Equation (4-65) becomes

$$
\frac{\partial}{\partial z} \int_{\varrho_0}^{\varrho_d} (\sigma_{ik} \hat{u}_k) \hat{n}_r r \, dr = -a (\sigma_{ik} \hat{u}_k) \hat{n}_r \bigg|_{r=a} \quad (4-66)
$$

The left hand side of Equation (4-66) refers to the spatial derivative of the average power through the cross-section while the right hand side refers to the average power through the side evaluated at the pile-soil interface. Substituting Equations (4-61) and (4-62) into Equation (4-66) gives

$$
\frac{\partial}{\partial z} \int_{\varrho_0}^{\varrho_d} \left( (\tau_{zz} \hat{u}_z) n_z + (\sigma_{zz} \hat{u}_z) n_z \right) r \, dr = -u \left( (\sigma_{ii} \hat{u}_i) n_i + (\tau_{ii} \hat{u}_i) n_i \right) \bigg|_{r=a} \quad (4-67)
$$

Substituting Equation (4-60) into (4-67) gives

$$
\frac{\partial}{\partial z} \left( \frac{\omega}{4i} e^{iz} \int_{\varrho_0}^{\varrho_d} \left[ \left( \tau_{zz} \hat{u}_z(r) - \tau_{xx}^*(r) \hat{u}_x(r) \right) n_z \right] \right) r \, dr
= \frac{\omega a}{4i} e^{iz} \left[ \left( \sigma_{ii} \hat{u}_i(r) - \sigma_{ii}^*(r) \hat{u}_i(r) \right) n_i \right] \bigg|_{r=a} \quad (4-68)
$$

Because the exponential is the only term that is a function of $z$, differentiating the left hand side of Equation (4-68) with respect to $z$ gives
\[
\frac{\omega \xi}{2i} e^{\frac{2i \xi}{z}} \int_{r=a}^{r=a} \left\{ \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) - \tau_{\alpha}^*(r) u_{\alpha}(r) \right] n_z + \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) - \sigma_{\alpha}^*(r) u_{\alpha}(r) \right] n_z \right\} r \, dr
\]

\[
= \frac{\omega a}{4i} e^{\frac{2i \xi}{z}} \left\{ \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) - \sigma_{\alpha}^*(r) u_{\alpha}(r) \right] n_r + \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) - \tau_{\alpha}^*(r) u_{\alpha}(r) \right] n_r \right\} \bigg|_{r=a} \tag{4-69}
\]

Rearranging Equation (4-69) gives

\[
\int_{r=a}^{r=a} \left\{ \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) - \tau_{\alpha}^*(r) u_{\alpha}(r) \right] + \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) - \sigma_{\alpha}^*(r) u_{\alpha}(r) \right] \right\} n_z \, dr
\]

\[
= \frac{a}{2 \xi} \left\{ \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) - \tau_{\alpha}^*(r) u_{\alpha}(r) \right] + \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) - \sigma_{\alpha}^*(r) u_{\alpha}(r) \right] \right\} n_r \bigg|_{r=a} \tag{4-70}
\]

The imaginary part of the product of two complex variables can be expressed as

\[
\text{Im} \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) \right] = \frac{\sigma_{\alpha}^*(r) u_{\alpha}^*(r) - \sigma_{\alpha}^*(r) u_{\alpha}(r)}{2i} \tag{4-71}
\]

Substituting Equation (4-71) into (4-70) gives

\[
\int_{r=a}^{r=a} \left\{ \text{Im} \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) \right] + \text{Im} \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) \right] \right\} n_z \, dr
\]

\[
= \frac{a}{2 \xi} \left\{ \text{Im} \left[ \sigma_{\alpha}^*(r) u_{\alpha}^*(r) \right] + \text{Im} \left[ \tau_{\alpha}^*(r) u_{\alpha}^*(r) \right] \right\} n_r \bigg|_{r=a} \tag{4-72}
\]
Equation (4-72) relates the power through a cross-section of the pile with the
power through the sides of the pile. The average power flowing through a cross-
section of the pile, at an arbitrary position along its axis, is given by

\[
\langle P_z \rangle = \oint_{S_a} \left[ \langle \tau_{rz} \hat{u}_r \rangle n_z + \langle \sigma_{zz} \hat{u}_z \rangle n_z \right] dA
\]

\[
= 2\pi \int_{r=0}^{r=a} \left[ \langle \tau_{rz} \hat{u}_r \rangle n_z + \langle \sigma_{zz} \hat{u}_z \rangle n_z \right] r \, \, dr
\]

\[
= \pi \omega \epsilon^{2jz} \int_{r=0}^{r=a} \left\{ \left[ \text{Im} [\tau_{rz} (r) u^*_r (r)] + \text{Im} [\sigma_{zz} (r) u^*_z (r)] \right] n_z \right\} r \, \, dr
\]  

(4-73)

Substituting Equation (4-72) into (4-73) gives

\[
\langle P_z \rangle = \frac{\pi \alpha \epsilon}{2 E} \epsilon^{2jz} \left\{ \text{Im} [\sigma_{zz} (r) u^*_r (r)] n_z + \text{Im} [\tau_{rz} (r) u^*_z (r)] n_z \right\} \bigg|_{r=a}
\]

(4-74)

The average power flowing through the cross-section of the pile must be
normalized to compare the displacement and stress distributions among any of the
guided wave modes. Equation (4-74) is a linear function of the stress and displacement
and, thus, the stress and displacement can be multiplied by any constant. Multiplying
the stress and displacement in Equation (4-74) by a real constant, \(g\), and selecting the
position \(z=0\), for simplicity to compare the power from different modes, the
normalized average power through the pile cross-section, $\langle \overline{P}_z \rangle$, is thus given by

$$\langle \overline{P}_z \rangle = -g^2 \frac{\pi \omega a}{2 \xi_i} \left\{ \text{Im}\left[ \sigma_{\tau}(r)u_\tau^*(r) \right] n_r + \text{Im}\left[ \tau_{\tau}(r)u_\tau^*(r) \right] n_r \right\}_{r=a}$$

(4-75)

The angular frequency and wave number is expressed in non-dimensional form by substituting Equation (4-16) into (4-75), giving

$$\langle \overline{P}_z \rangle = -g^2 \frac{\pi \Omega c_r a}{2(\xi, a)} \left\{ \text{Im}\left[ \sigma_{\tau}(r)u_\tau^*(r) \right] n_r + \text{Im}\left[ \tau_{\tau}(r)u_\tau^*(r) \right] n_r \right\}_{r=a}$$

(4-76)

The normalized average power flowing through the pile cross-section, $\langle \overline{P}_z \rangle$, is chosen to be unity and Equation (4-76) becomes

$$\left| -g^2 \frac{\pi \Omega c_r a}{2(\xi, a)} \text{Im}\left[ \sigma_{\tau}(r)u_\tau^*(r) + \tau_{\tau}(r)u_\tau^*(r) \right] n_r \right|_{r=a} = 1$$

(4-77)

The normalizing constant, $g$, is then given by

$$g = \sqrt{\frac{2(\xi, a)}{\pi \Omega c_r a \text{Im}\left[ \sigma_{\tau}(r)u_\tau^*(r) + \tau_{\tau}(r)u_\tau^*(r) \right] n_r |_{r=a}}}$$

(4-78)
Thus, the normalized displacements in the pile and soil with respect to unit power flowing through a cross-section of the pile are then obtained by multiplying Equations (4-39) to (4-42) by \( g \). The normalized displacements from different modes can then be compared directly to evaluate the relative amplitudes of the axial and radial displacement components.

### 4.7 GROUP VELOCITY

The previous sections have shown that there are an infinite number of longitudinal, torsional and flexural solution modes for wave propagation in the pile. The modes may be visualized as wave packets, or groups, and they propagate at a velocity that may be different from the propagation velocities of the individual solution modes. Consider two propagating harmonic waves of equal amplitude but with slightly different angular frequencies and wave numbers

\[
u_1(z,t) = A\cos(\omega_1 t - \xi_1 z)
\]

\[
u_2(z,t) = A\cos(\omega_2 t - \xi_2 z)
\]

The sum of their displacements is given by

\[
u(z,t) = A\cos(\omega_1 t - \xi_1 z) + A\cos(\omega_2 t - \xi_2 z)
\]
Since the frequencies and wave number are slightly different, the following substitutions are introduced

\[
\begin{align*}
\omega_1 &= \omega + \Delta \omega \\
\omega_2 &= \omega - \Delta \omega \\
\xi_1 &= \xi + \Delta \xi \\
\xi_2 &= \xi - \Delta \xi
\end{align*}
\] (4-82)

where \(\omega \gg \Delta \omega\) and \(\xi \gg \Delta \xi\)

Substituting Equation (4-82) into (4-81) gives

\[
u(z,t) = A \cos(\omega t + \Delta \omega t - \xi z - \Delta \xi z) + A \cos(\omega t - \Delta \omega t - \xi z + \Delta \xi z) \\
= 2A \cos(\omega t - \xi z) \cos(\Delta \omega t - \Delta \xi z)
\] (4-83)

The sum of the two harmonic cosine functions results in the product of two harmonic cosine functions of different frequencies and wave numbers. The first cosine term, \(\cos(\omega t - \xi z)\) is the high frequency term propagating at an average phase velocity, \(c_m = \omega / \xi\). The second cosine term, \(\cos(\Delta \omega t - \Delta \xi z)\) is the lower frequency term that propagates at the group velocity, \(c_g = \Delta \omega / \Delta \xi\). Thus, the net effect is a high frequency carrier wave modulated by a low frequency envelope. This envelope or beat pattern propagates with the group velocity.
Generalizing the above for a number of solutions, it may be shown that the group velocity is given by,

$$c_x = \frac{d\omega}{d\xi} \quad (4-84)$$

Using the relation $\omega = \xi c$, the group velocity may also be expressed as

$$c_x = \frac{d(\xi c)}{d\xi} = c + \xi \frac{dc}{d\xi} \quad (4-85)$$

Alternatively, using the relation, $\xi = \frac{2\pi}{\lambda}$, the group velocity may also be expressed as,

$$c_x = c - \frac{\lambda}{d\lambda} \frac{dc}{d\lambda} \quad (4-86)$$

The advantage of plotting the dispersion curves on the frequency spectrum plot shown in Figure 4-2 is the direct interpretation of the gradient and secant at any point on the curve. The solid lines at points A and B denote the gradient while the dotted lines denote the secant. The gradient, $(d\omega/d\xi)$, to the curve at any point on the branch corresponds to the group velocity and the secant, $(\omega/\xi)$, corresponds to the
phase velocity. The gradient at point A is less than the secant and therefore, $c_g < c$ while the situation is reversed at point B and $c_g > c$. In the non-dimensional frequency, $\Omega$, versus non-dimensional wave number, $\xi/a$ plot, the gradient and secant correspond to the non-dimensional group velocity and non-dimensional phase velocity, respectively, normalized by the shear wave velocity in the pile.

Figure 4-2   Frequency Spectrum Example

4.8 SUMMARY

The frequency equation for longitudinal modes was presented in a non-dimensional form by a $4 \times 4$ determinant. Non-dimensional variables, which constitute the components of the determinant, were defined in terms of the Poisson's ratios, shear moduli and densities of the pile and soil. The frequency equation thus represents a
transcendental relationship between the non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$.

The axial and radial displacements and stresses in the pile and soil were derived. For displacements or stresses of any mode to be compared directly, a normalizing constant was derived. This constant set the energy flux or power of any mode through a cross-section of the pile to be unity. The phase and group velocities of guided wave modes were defined and the determinations of these velocities from the dispersion curves were illustrated.
5.1 INTRODUCTION

This chapter presents a numerical evaluation of the frequency equation for longitudinal modes in a cylindrical pile embedded in soil. The frequency equation developed in Chapter 4, represented by a non-dimensional 4x4 determinant, involves a transcendental relationship between the non-dimensional frequency, \( \Omega \), and non-dimensional wave number, \( \xi_a \). A computer code for the embedded pile is written to solve numerically this transcendental relationship between \( \Omega \) and \( \xi_a \). A symbolic mathematical package that provides built-in commands to simplify the programming is used. A code is also developed to plot the longitudinal branches for the free pile. Typical values for elastic constants and material properties for the concrete pile and soil are considered. The elastic soil parameters investigated represented soils that varied in consistency from soft/loose to hard/dense, and thus represent a wide range of natural soils.

The first five branches of the longitudinal modes in an embedded pile are determined. These branches are comprised of complex wave numbers in the real frequency domain. The frequency equation and the embedded pile code are verified by
considering the limit as the properties of the surrounding soil approach that of air, yielding the well-known solutions to the free rod. The effects of shear modulus and density ratios on the dispersion curves are studied.

5.2 COMPUTER CODE TO GENERATE DISPERSION CURVES

5.2.1 Outline of Mathematica Code

A symbolic mathematical software package was used to solve the system of linear algebraic equations appearing as the 4x4 determinant in the frequency equation. The availability of specific routines dealing with the special functions, e.g. the Bessel and Modified Bessel functions, in Mathematica (Wolfram. 1996), prompted its use in carrying out the numerical computations.

The code consists of two parts. The first part deals with the calculation of the roots of the frequency equation. The relationship between frequency and the wavenumber was established, giving the frequency spectrum. The frequency spectrum is composed of an infinite number of branches in the real frequency-complex wavenumber domain. The second part of the code deals with the displacement calculations of given modes. It involves the determining the coefficient matrix, establishing the normalizing constant from energy considerations, and matching the displacements at the pile-soil interface. The Mathematica code is presented in the Appendix.
5.2.2 Concrete and Soil Parameters

The components of the 4x4 determinant in the frequency equation for longitudinal modes involve various concrete and soil parameters. These include the shear modulus, \( \mu \), Poisson's ratio, \( \nu \), density, \( \rho \), bulk longitudinal wave velocity, \( c_l \), and bulk transverse wave velocity, \( c_t \). The values for the concrete parameters were selected assuming typical concrete mix designs for drilled shafts. A concrete compressive strength of 40 MPa was assumed and the corresponding Young's modulus was computed to be approximately 30 GPa using the following empirical relationship from the American Concrete Institute Section 8.5.1,

\[
E = 1.5 \times 10^5 \sqrt{f_{cu}}
\]  

(5-1)

where \( E \) is the Young's modulus in kPa and \( f_{cu} \) is the cube compressive strength in kPa.

The shear modulus was calculated to be 13 GPa assuming a Poisson's ratio of 0.18. The density of concrete was taken as 2400 kg/m\(^3\). The bulk shear wave velocity was calculated using Equation (4-30) to be 2325 m/s and the bulk longitudinal wave velocity as 3725 m/s using Equation (4-21).
For the purpose of this study, the pile is assumed to be embedded in soil, which ranges from soft/loose to hard/dense soils. Bulk shear wave velocities of 150 m/s and 380 m/s were assigned to the low and high stiffness soils, respectively, so that a range of conditions normally encountered could be evaluated. Assuming densities of 1800 kg/m$^3$ and 2000 kg/m$^3$ for the low and high stiffness soils respectively, their corresponding shear moduli were calculated to be 40 MPa and 290 MPa, respectively. The Poisson's ratio for the soil was taken as 0.3 and the bulk longitudinal wave velocities were calculated as 280 m/s and 710 m/s for the low and high stiffness soils, respectively. The parameters used in the computations are summarized in Table 5-1 shown below.

Table 5-1 Assumed Concrete and Soil Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Concrete</th>
<th>Soft/Loose Soil</th>
<th>Hard/Dense Soils</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson's Ratio</td>
<td>0.18</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Density (kg/m$^3$)</td>
<td>2400</td>
<td>1800</td>
<td>2000</td>
</tr>
<tr>
<td>Young's Modulus (MPa)</td>
<td>30000</td>
<td>105</td>
<td>755</td>
</tr>
<tr>
<td>Shear Modulus (MPa)</td>
<td>13000</td>
<td>40</td>
<td>290</td>
</tr>
<tr>
<td>Shear Wave Velocity (m/s)</td>
<td>2325</td>
<td>150</td>
<td>380</td>
</tr>
<tr>
<td>Longitudinal Wave Velocity (m/s)</td>
<td>3725</td>
<td>280</td>
<td>710</td>
</tr>
<tr>
<td>Density Ratio</td>
<td>-</td>
<td>1.3</td>
<td>1.2</td>
</tr>
<tr>
<td>Shear Modulus Ratio</td>
<td>-</td>
<td>325</td>
<td>45</td>
</tr>
</tbody>
</table>
5.2.3 Numerical Analysis

5.2.3.1 Initial Roots

In Section 4.2, it was shown that the dispersion curves for longitudinal modes in an embedded pile can be obtained by solving the frequency equation given by Equation (4-4). When expanded, the 4x4 determinant in the equation results in a transcendental relationship between the non-dimensional frequency, $\Omega$, and the non-dimensional wave number, $\xi_a$. There is no analytical or closed-form solution, and therefore, the equation has to be solved numerically.

Mathematica was used to solve the frequency equation symbolically with the use of the “FindRoot” command. There are several variations of the “FindRoot” command but in all cases, it requires at least one initial guess that is close to the solution. If only one initial guess is given, “FindRoot” uses Newton’s method. If symbolic derivatives of the equation cannot be found, “FindRoot” uses a variant of the secant method and two initial guesses are required (Wolfram, 1996). In carrying out the computations, it was found that both variations of “FindRoot” would work but the secant version was computationally faster and was used for the calculations.

Zemanek (1960) showed that there are a doubly-infinite number of solutions to the frequency equation of the free rod. For longitudinal modes alone, there is an
infinite number of branches as shown in Figure 2-7. On a plot of frequency against wave number, the real branches of the free pile intersect the frequency axis. The point of intersection of a branch and the frequency axis is termed the cut-off frequency of that particular branch.

The cut-off frequencies for the free pile case are readily determined. By putting the non-dimensional wave number, $\xi a = 0$ into Equation (2-10), the frequency equation for the free pile simplifies into a product of two expressions, each containing Bessel functions. By solving for the frequency in these expressions, the cut-off frequencies for the axial-shear and radial-shear modes are obtained. The cut-off frequencies can then be used as initial guesses in “FindRoot” to obtain the first root on a given branch.

In contrast, by putting $\xi a = Y = 0$ in the embedded pile frequency equation represented by the 4x4 determinant in Equation (4-4), a product of two 2x2 sub-determinants results. These sub-determinants contain Bessel and Hankel functions and form complex expressions when expanded. The cut-off frequencies for the longitudinal modes in this case are not easily calculated. In view of this difficulty, the cut-off frequencies from the free pile case were used as the initial guesses for the embedded pile code.
5.2.3.2 Curve Tracing

Several methods may be used to plot the infinite number of branches of the frequency spectrum. One approach is to pick a value for frequency, starting near zero, and scanning the domain of wave numbers to obtain roots that satisfy the frequency equation. This is repeated for increasing values of frequency. The roots are then connected to form the various branches of the dispersion curves.

The second approach is to pick a value for wave number and scan the domain of frequencies for roots of the frequency equation. However, this method works well only if real wave numbers are expected. For complex wave numbers, there are now three variables, the real part of the wave number, the imaginary part of the wave number and the frequency. The approach to solve this problem is to start by fixing the real part of the wave number, and for incremental values of the imaginary part of the wave number, the domain of frequencies is scanned for roots. The process is then repeated for increasing values of the real part of the wave number so that the frequency-wave number relationship is defined for the range of interest. The roots are then connected to form the various branches of the dispersion curves.

The third approach to solve the problem is to start by fixing the imaginary part of the wave number and cycling through the real part of the wave number and scanning the domain of frequencies for roots. This process is then repeated for
increasing values of the imaginary part of the wave number so that the frequency spectrum is covered. The roots are again connected to form the various branches of the dispersion curves. The second and third methods are computationally intensive, and therefore are unsuitable.

Figure 5-1 illustrates the scheme used in tracing the longitudinal branches. Because the frequency is real and the wave number is complex, the frequency is varied in small equal increments and the wave number is determined numerically using “FindRoot”. As explained in the last section, the first root on the branch is determined using the cut-off frequency as the initial guess. The first root then serves as the initial guess in “FindRoot” to determine the second root. The third root is determined by using the second root as the initial guess in “FindRoot” again. Using an iterative process, the procedure keeps the solutions generated on the same branch as long as small increments of frequency are used. Because complex wave numbers are expected, the frequency is progressively varied and the wave number is determined numerically.

5.2.3.3 Step Sizes

The only two unknowns in the frequency equation after specifying all the material variables are the frequency and the wave number. Recall that there are an infinite number of solutions to the frequency equation. For a given value of the
frequency, the "FindRoot" command returns a solution for the wave number based on the initial roots. A "Do Loop" was used to run the FINDROOT routine successively with increasing values of frequency. This method enabled the solution to march along the curve and thus obtaining the branch. The step size here refers to the increment in frequency and not to the step size in the FINDROOT routine. It was found that large frequency increments would cause the solutions to jump from one branch to another. It was found that using a step size of 0.01 to 0.1 for the non-dimensional frequency resulted in a stable algorithm.
5.3 DISPERSION CURVES FOR LONGITUDINAL MODES IN AN EMBEDDED PILE

Dispersion curves for five branches of longitudinal modes were generated using the numerical solution techniques described in Section 5.2.3. The soil surrounding the pile was modeled as having either low or high stiffness, as reflected by the shear wave velocity of the soil. The first five longitudinal branches for both the soft/loose and the hard/dense soils are shown in Figures 5-2, 5-3 and 5-4.

Figure 5-2 shows the non-dimensional frequency plotted against the real part of the non-dimensional wave number for both values of shear modulus ratios used in this study. There is a negligible difference between the two curves and for practical purposes, they are the same. Thus, the real part of the wave number is not affected by changes in the shear modulus ratios investigated herein.

The L(0,1) branch starts from the origin and has a zero cut-off frequency. At high frequencies it becomes asymptotic to the Rayleigh wave velocity. The L(0,2) and L(0,4) branches do not intersect the $\Omega - \xi_a$ plane and therefore do not have cut-off frequencies. These branches extend into the imaginary wave number plane. The L(0,3) and L(0,5) branches intersect the $\Omega - \xi_a$ plane but their cut-off frequencies are different from their corresponding cut-off frequencies of the free-standing pile. All branches except the L(0,1) branch, are asymptotic to the bulk shear wave velocity at high frequencies.
Figure 5-2  First Five Branches of Longitudinal Modes in an Embedded Pile in $\Omega - \xi_n a$ Space for Soft/Loose Soil ($\mu_p/\mu_s = 325$) and Hard/Dense Soil ($\mu_p/\mu_s = 45$)
Figure 5-3 First Five Branches of Longitudinal Modes in an Embedded Pile in $\Omega - \xi, a$ Space for Soft/Loose Soil ($\mu_p/\mu_s = 325$)
Figure 5-4  First Five Branches of Longitudinal Modes in an Embedded Pile in $\Omega - \xi a$ Space for Hard/Dense Soil ($\mu_p/\mu_s = 45$)
Figure 5-3 and Figure 5-4 show the non-dimensional frequency plotted against the imaginary part of the non-dimensional wave number for the two limiting shear modulus ratios. Except for the L(0,1) branch, the branches have large imaginary wave numbers at lower frequencies with decreasing imaginary wave numbers at higher frequencies. The L(0,1) branch starts with a small imaginary component of the wave number at zero frequency but becomes extremely large as the frequency increases. The L(0,2) branch has the lowest imaginary part of the wave number compared to all other branches for non-dimensional frequencies above 10.

Comparing corresponding branches in Figure 5-3 and Figure 5-4, the shapes of the curves are generally the same. However, the imaginary part of the wave number of the hard/dense soil is larger than that of the soft/loose soil for every branch. It will be shown in Section 6.3 that the imaginary part of the wave number corresponds to the attenuation of the guided waves due to leakage. This implies that the attenuation of waves in embedded piles is higher in hard/dense soils than in soft/loose soils.

5.4 VERIFICATION OF DISPERSION CURVES

It is imperative that the theoretical formulation and the numerical solutions for the dispersion curves are verified in some way. In the current literature, there is no published work on the dispersion curves for embedded piles in soil, which could be used to corroborate these results. In the following sections, two approaches are used to
verify the dispersion curves. The first approach examines the frequency equation of the embedded pile while the second approach analyzes the embedded pile code.

5.4.1 Verification of Frequency Equation

The cylindrical pile embedded in soil may be viewed as a composite problem whereby the concrete pile is the inner material and the surrounding soil is the outer material. If the soil is replaced by air, this effectively leaves only the inner material and the embedded pile acts as a free pile. Hence, the embedded pile becomes free in the limit as the properties of the outer material approach that of air. Thus, in this limit, the frequency equation for the embedded pile should decompose into the well-known frequency equation for a free pile or rod.

The frequency equation for longitudinal modes in an embedded pile is represented by a $4 \times 4$ determinant as shown in Section 4.2, and reintroduced as Equation (5-2). In contrast, the frequency equation of the free pile or rod is represented by a $2 \times 2$ determinant (Graff, 1975), which may be expanded to yield the well-known Pochhammer-Chree Frequency Equation. The material parameters of the pile and soil which appear in Equation (5-2) are the Lamé constants $\lambda$, and $\mu$. The latter is the shear modulus and both constants may be expressed in terms of other elastic constants like Young's modulus (E) and Poisson's ratio ($\nu$). The properties of
\[
\begin{vmatrix}
-\nu Z_i(V) & \nu Z_i(X) & WH_i^{(3)}(V) & -\nu H_i^{(3)}(U) \\
-\nu Z_o(V) & -XZ_o(X) & \nu H_o^{(3)}(W) & UH_o^{(3)}(U) \\
-\frac{1}{\mu_p} \left( \nu^2 + Y^2 \right) + 2Y^2 \right] Z_o(V) + 2\nu Z_i(V) & 2Y[XZ_o(X) - Z_i(X)] & \frac{1}{\mu_p} \left( \nu^2 + Y^2 \right) + 2\frac{\nu}{\mu_p} W^2 \right] H_o^{(3)}(W) - 2\frac{\nu}{\mu_p} WH_i^{(3)}(V) & -2\frac{\mu_s}{\mu_p} Y[UH_o^{(3)}(U) - H_i^{(3)}(U)] \\
2Y\nu Z_i(V) & -[Y^2 - X^2] Z_i(X) & -2\frac{\mu_s}{\mu_p} YWH_i^{(3)}(W) & \frac{\mu_s}{\mu_p} [Y^2 - U^2] H_i^{(3)}(U) \\
\end{vmatrix}
= 0 \quad (5-2)
\]

\[
\begin{vmatrix}
-\nu Z_i(V) & \nu Z_i(X) & WH_i^{(3)}(V) & -\nu H_i^{(3)}(U) \\
-\nu Z_o(V) & -XZ_o(X) & \nu H_o^{(3)}(W) & UH_o^{(3)}(U) \\
-\frac{1}{\mu_p} \left( \nu^2 + Y^2 \right) + 2Y^2 \right] Z_o(V) + 2\nu Z_i(V) & 2Y[XZ_o(X) - Z_i(X)] & 0 & 0 \\
2Y\nu Z_i(V) & -[Y^2 - X^2] Z_i(X) & 0 & 0 \\
\end{vmatrix}
= 0 \quad (5-3)
\]
the surrounding soil approach that of air when $\lambda_i$ and $\mu_i$ approach zero. Thus, the ratios, $\lambda_i/\mu_p \to 0$ and $\mu_i/\mu_p \to u$ in Equation (5-2), and the 4x4 determinant of the embedded pile in the limiting case is shown in Equation (5-3). This determinant can then be decomposed into a product of sub-determinants shown in Equation (5-4).

\[
\begin{vmatrix}
- \frac{\lambda_p}{\mu_p} (\nu^2 + \gamma^2) + 2\nu^2 & Z_0(\nu) + 2\nu Z_i(\nu) & 2Y[XZ_o(X) - Z_i(X)] \\
2YVZ_i(\nu) & - (\gamma^2 - \xi^2)Z_i(\xi) & \\
x & WH^{(2)}(W) & - YH^{(2)}(U) \\
YH^{(2)}(W) & UH^{(2)}(U) &
\end{vmatrix} = 0 \quad (5-4)
\]

The first determinant in Equation (5-4) can be expanded out to give

\[
\begin{align*}
\left\{ - \frac{\lambda_p}{\mu_p} (\nu^2 + \gamma^2) + 2\nu^2 \right\} Z_0(\nu) + 2\nu Z_i(\nu) - \left(\gamma^2 - \xi^2\right)Z_i(\xi) \\
- \{2YVZ_i(\nu)\} \{2Y[XZ_o(X) - Z_i(X)]\} = 0
\end{align*} \quad (5-5)
\]

Substituting Equations (4-24) and (4-25) into Equation (4-32) gives the ratio of the Lamé parameters as

\[
\frac{\lambda_p}{\mu_p} = \frac{X^2 - Y^2 - 2\nu^2}{\nu^2 + Y^2} \quad (5-6)
\]
Substituting Equation (5-6) into (5-5) and simplifying gives

\[-(X^2 - Y^2)Z_o(V)Z_i(X) + 2V(X^2 + Y^2)Z_i(V)Z_o(X) - 4VXY^2 Z_i(V) Z_o(X) = 0\]  \hspace{1cm} (5-7)

Substitute back \( V = \alpha_p a \), \( X = \beta_p a \) and \( Y = \xi a \) gives

\[-(\beta_p^2 - \xi^2)^2 Z_o(\alpha_p a)Z_i(\beta_p a) + \frac{2\alpha_p}{a}(\beta_p^2 + \xi^2)Z_i(\alpha_p a)Z_o(\beta_p a) - 4\xi^2 \alpha_p \beta_p Z_i(\alpha_p a)Z_o(\beta_p a) = 0\]  \hspace{1cm} (5-8)

which is the well-known Pochhammer-Chree Frequency Equation. Therefore, the terms in the determinant for longitudinal modes in the embedded pile case decompose to the Pochhammer-Chree frequency equation for a free rod in the limit, as the soil surrounding the pile becomes “air-like.” This decomposition helps substantiate the theoretical formulation presented in Chapter 3.

### 5.4.2 Verification of the Embedded Pile Code

The embedded pile code developed in Section 5.2.1 is based on the frequency equation shown in Equation (5-2) and the resulting dispersion curves were presented in Section 5.3. As explained in Section 5.4.1, in the limit as the properties of the outer material approach that of air, the embedded pile becomes free, i.e. a “free” embedded pile. Thus, using appropriate values for the shear modulus and density of the outer
material, the embedded pile code should generate dispersion curves for longitudinal modes that match those in a free pile or rod.

The dispersion curves for longitudinal modes in free rods are well documented in the literature. The dispersion curves published by Zemanek (1960), for longitudinal and flexural modes in a free rod with Poisson's ratio, $\nu = 0.3317$, are shown in Figures 2.7 and 2.8. The Poisson's ratio for concrete used in this study is 0.18. Thus, the dispersion curves for a free concrete pile with $\nu = 0.18$ are required to enable comparison with the dispersion curves generated by the embedded pile code in the limiting case where the properties of the surrounding soil approach that of air.

A computer code was written in Mathematica to solve the Pochhammer-Chree frequency equation shown in Equation (5-8). The free pile Mathematica code is presented in the Appendix. The dispersion curves were first generated using Poisson's ratio, $\nu = 0.3317$. The first five branches of the longitudinal modes were plotted and it was verified that these curves matched the curves published by Zemanek (1960), shown in Figure 2.7. This procedure demonstrated that the free pile code produced accurate results. The Poisson's ratio was then changed to 0.18 to represent the concrete pile. The first five branches of the longitudinal modes in a free concrete pile were generated and shown in Figure 5-5.
Figure 5-5  First Five Branches of Longitudinal Modes in a Free-Standing Pile, $v = 0.18$
The next step is to generate the dispersion curves for the “free” embedded pile using the embedded pile code with the parameters of the soil adjusted to represent that of air. The shear moduli and densities of the pile and soil are expressed as ratios in the embedded pile code. For the case of typical soft/loose soil, the shear modulus ratio, \( \mu_p / \mu_s = 325 \) and the density ratio, \( \rho_p / \rho_s = 1.2 \) were used. When the soft/loose soil is replaced by material with properties, which approach that of air, these ratios would substantially increase. The shear modulus of air is zero and therefore the shear modulus ratio would be infinity. To approximate the properties of air, an extremely large value is assigned for the shear modulus ratio, \( \mu_p / \mu_s = 325 \times 10^6 \). The density of air is approximately \( 1 \text{ kg/m}^3 \), and the density ratio is taken as \( \rho_p / \rho_s = 2400 \). These ratios thus represent the case where the embedded pile acts as a “free” pile.

Figure 5-6 and Figure 5-7 show the dispersion curves for the “free” embedded pile. Figure 5-6 shows the frequency plotted against the real part of the wave number for the first five branches, while Figure 5-7 shows the frequency plotted against the imaginary part of the wave number. By superimposing Figures 5-5 and 5-6, it is clear that the dispersion curves for a free pile from the free pile code and the dispersion curves for the “free” embedded pile from the embedded pile code match exactly. The imaginary part of the wave number for all the five branches from the embedded pile code, shown in Figure 5-7, are extremely small and can be considered to be zero. Thus, as expected, the embedded pile code generates the dispersion curves in a free
Figure 5-6  First Five Branches of Longitudinal Modes in a “Free” Embedded Pile in $\Omega - \xi, \alpha$ Plane with $\mu_p/\mu_s = 325 \times 10^6$ and $\rho_p/\rho_s = 2400$
Figure 5-7  First Five Branches of Longitudinal Modes in a “Free” Embedded Pile
in $\Omega-\xi a$ Plane with $\mu_p/\mu_s = 325 \times 10^6$ and $\rho_p/\rho_s = 2400$
pile when the properties of the soil surrounding the pile approach that of air. This validates the embedded pile code.

5.4.3 Effect of Shear Modulus and Density Ratios

Because there is no published work on dispersion curves for piles embedded in soil, the behavior of the embedded pile modes must be verified by examining the transition of the free pile case to the embedded pile case. In the free pile case, the Poisson's ratio is the only material variable that affects the dispersion curves. In the embedded pile case, the shear moduli and densities of the pile and soil also affect the behavior. The shear modulus and density of the pile and soil are expressed as the ratios $\mu_p/\mu_s$ and $\rho_p/\rho_s$, respectively.

The transition from the free pile to the embedded pile is explored by considering the free pile case using the embedded pile code, i.e. the "free" embedded pile, and changing the shear modulus and density ratios, independently in stages until the appropriate values are reached. For the free pile case, shear modulus and density ratios of $325 \times 10^6$ and $2.4 \times 10^3$, respectively, were used in the computations. For the embedded pile case, the soft/loose soil condition was modeled with a shear modulus ratio of 325 and density ratio of 1.2. Three stages were used in moving from the free pile to the embedded pile case. Each stage involved a change in the shear modulus of two orders of magnitude or a change in the density ratio of an order of magnitude.
Figure 5-8 and Figure 5-9 show the frequency of the L(0,1) branch plotted against the real and imaginary parts of the wave number at a density ratio of \( \rho_p/\rho_s = 2.4 \times 10^3 \). Four values of the shear modulus ratio are plotted in these figures. These range from the free pile case with \( \mu_p/\mu_s = 325 \times 10^6 \) to the embedded pile case with \( \mu_p/\mu_s = 325 \). It is evident from Figure 5-8 that the real part of the wave number is unaffected by changes in the shear modulus of the soil. All of the four cases plot on the same curve showing that the real part of the wave number is essentially that of the free rod. It is interesting to note that this occurs although the shear modulus ratio is changed by six orders of magnitude.

However, Figure 5-9 shows that the imaginary part of the wave number is strongly affected by changes in the shear modulus ratio. The dashed line represents the free pile case and it is virtually zero. As the shear modulus ratio decreases, which implies that the soil becomes relatively stiffer, the imaginary component of the wave number increases. The stiffer the soil, the larger the imaginary part of the wave number for a given density ratio. The significance of the imaginary part will be explained in Chapter 6.

The effect of varying the density ratio while keeping the shear modulus ratio constant was also investigated. The shear modulus ratio was fixed at \( \mu_p/\mu_s = 325 \times 10^6 \), i.e. modeling the soil as air, and the density ratio was varied from
Figure 5-8  L(0,1) Branches for an Embedded Pile in $\Omega - \xi, a$ Plane with Density Ratio $\rho_p/\rho_s = 2.4 \times 10^1$ for Varying Shear Modulus Ratios
Figure 5-9  
L(0,1) Branches for an Embedded Pile in $\Omega - \xi, a$ Plane with Density Ratio $\rho_p/\rho_s = 2.4 \times 10^3$ for Varying Shear Modulus Ratios
the free pile case of $\rho_p/\rho_s = 2.4 \times 10^1$ to the embedded pile case of $\rho_p/\rho_s = 1.2$ in three decrements. The frequency is plotted against the real and imaginary parts of the wave number for the L(0,1) branch in Figure 5-10 and Figure 5-11, respectively.

It is clear that the trends are the same as those just described for the shear modulus. The free pile case is represented by the dashed line where $\mu_p/\mu_s = 325 \times 10^6$ and $\rho_p/\rho_s = 2.4 \times 10^1$. Figure 5-10 shows that the real part of the wave number for all the four density ratios is exactly the same as that of the free pile, even though the density ratio was changed by two orders of magnitude. Conversely, the imaginary part of the wave number changes with the density ratio. As the density of the surrounding soil increases, the density ratio decreases and the imaginary part of the wave number increases. The denser the soil, the larger the imaginary part of the wave number for a given shear modulus ratio.

These results show that the real part of the wave number of the embedded pile is not affected by changes in the shear modulus or density of the surrounding soil and corresponds to the wave number of the free-standing pile. In contrast, the imaginary part of the wave number is affected by changes in both the shear modulus and density ratios. For the range of soil conditions considered, the imaginary part the wave number is more dependent on the shear modulus ratio. In general, the imaginary component of the wave number increases with the stiffness and density of the soil.
Figure 5-10  L(0,1) Branches for an Embedded Pile in $\Omega - \xi \alpha$ Plane with Shear Modulus Ratio $\mu_p / \mu_s = 325 \times 10^4$ for Varying Density Ratios
Figure 5-11  L(0,1) Branches for an Embedded Pile in \( \Omega - \xi, a \) Plane with Shear Modulus Ratio \( \mu_p/\mu_s = 325 \times 10^6 \) for Varying Density Ratios
5.5 SUMMARY

A numerical code was developed to plot the dispersion curves for longitudinal modes in an embedded pile in soil. The first five branches of longitudinal modes were plotted for a realistic range of soil conditions that varied from soft/loose, denoted by a shear modulus ratio of 325, to hard/dense soils, denoted by a shear modulus ratio of 45. The frequency versus real part of the wave number plots are virtually the same for both limiting cases. The imaginary part of the wave number of the hard/dense soil was larger than that of the soft/loose soil. All the branches showed large imaginary values at low frequencies and progressively smaller values at high frequencies except the L(0,1) branch that showed an opposite trend.

The code was verified by analytically considering the limiting case of the 4x4 determinant appearing in the frequency equation, and, numerically by comparing free pile dispersion curves computed from the embedded pile code and free pile code. In the free pile case, the wave number is real while in the embedded pile case, the wave number is complex. Sensitivity analysis carried out showed that the real part of the wave number was insensitive to changes in the shear modulus ratio and density ratio, but the imaginary part of the wave number was affected by changes in these quantities, particularly the shear modulus ratio. The real part of the wave number of the embedded pile was the same as the wave number of the corresponding free pile,
except near the cut-off frequencies. The smaller the shear modulus and density ratios, and hence the stiffer the soils, the larger the imaginary part of the wave numbers.
6.1 INTRODUCTION

In this chapter, the wave numbers of guided wave modes in an embedded cylindrical concrete pile are investigated. The effects of the relative magnitudes of the shear wave velocities in the pile and soil on the wave number solutions are studied. Real wave numbers correspond to free or non-attenuating modes, while complex wave numbers correspond to leaky or attenuating modes. In the case of a deep foundation, the loss of wave energy into the surrounding soil results in the attenuation of waves propagating along the pile. A method of quantifying the attenuation of the waves is derived. The attenuation is defined in nepers and decibels.

The variations of the non-dimensional phase and group velocities with non-dimensional frequency for the first five longitudinal branches are presented. Fifteen guided wave modes from all the five branches are selected for evaluation based on low imaginary wave numbers, as well as maximum and minimum group velocities. The distribution of normalized power and the axial and radial displacement profiles in the pile for the seventeen modes are presented. Guided wave modes that have favorable characteristics for propagation are identified.
6.2 WAVE NUMBER

In the formulation of the embedded pile problem, the boundary conditions require that the generated stress waves propagate axially along the pile and radially outwards into the soil. The choice of the type of special function, i.e. the Hankel or Bessel function, to describe wave propagation outwards and into the soil, depends on whether the special function arguments in the soil, $\alpha r$ and $\beta r$, are real, imaginary or complex. As discussed in Section 6.2.1, these arguments depend on the material properties of the pile and soil and the wave number. The wave number is determined numerically and can be real, imaginary or complex. Real wave numbers correspond to free or non-attenuating modes. Imaginary wave numbers correspond to evanescent or non-propagating modes. Complex wave numbers correspond to propagating modes that attenuate with distance, or leaky modes.

Thurston (1978) examined the propagation of elastic waves in rods with infinite thickness cladding and showed that the material properties, specifically the ratio of the shear wave velocities of the rod and cladding determine whether free or leaky mode solutions are obtained. It will be shown in this section that free modes are obtained when the shear wave velocity in the pile, $c_{r_p}$ is less than the shear wave velocity in the soil, $c_{r_s}$. In the case of deep foundations made of concrete, only leaky modes are
obtained because the shear wave velocity in the pile is higher than the shear wave velocity in the soil.

The special function arguments, $\alpha_i$ and $\beta_i$, are obtained from substitutions made in the partial differential equations involving the potential functions for the soil, as shown in Equations (3-23) and (3-32), and are re-introduced as

$$\alpha_i^2 = \frac{\omega^2}{c_{i_s}^2 - \xi^2} \quad (6-1)$$

$$\beta_i^2 = \frac{\omega^2}{c_{i_s}^2 - \xi^2} \quad (6-2)$$

The angular frequency, $\omega$, wave number, $\xi$ and the phase velocity, $c$ of a wave are related by the following expression

$$\omega = \xi c \quad (6-3)$$

Substituting Equation (6-3) into Equations (6-1) and (6-2) gives

$$\alpha_i^2 = \xi^2 \left( \frac{c_{i_s}^2}{c_{i_s}^2 - 1} \right) \quad (6-4)$$
\[ \beta^2_i = \xi^2 \left( \frac{c_i^2}{c_{r_i}^2} - 1 \right) \]  \hspace{1cm} (6-5)

6.2.1 Shear Wave Velocity in the Pile Greater than Shear Wave Velocity in the Soil

In deep foundations, the relative magnitudes of the shear and longitudinal wave velocities in the pile and soil are typically ordered as \( c_{l_p} > c_{r_p} > c_{l_s} > c_{r_s} \). When the phase velocity, \( c \), of modes in the pile are higher than the shear and longitudinal wave velocities in the soil, the ratios, \( c^2/c_{l_s}^2 \) and \( c^2/c_{r_s}^2 \) in Equations (6-4) and (6-5) respectively, are larger than unity and the bracketed terms are thus positive.

Assuming the wave number is real, then \( \alpha^2 \) and \( \beta^2 \) are both positive. Hence, the special function arguments \( \alpha r \) and \( \beta r \) are real. The appropriate function that satisfies the condition of outwards propagating waves in the soil when multiplied by the time factor, \( e^{-i\omega t} \), is therefore the Hankel function of the second kind, \( H_n^{(2)} \). Because the outgoing waves in the soil are propagating radially to infinity, the selected function must decay at large radius, \( r \). The asymptotic expression for \( H_n^{(2)} \) with large arguments is given by

\[ H_n^{(2)}(x) \sim \frac{2}{\sqrt{\pi x}} e^{-\frac{i}{4} x \left( 1 + \frac{n\pi}{x} \right)} \]  \hspace{1cm} (6-6)
Replacing $x$ by $\beta, r$ and multiplying Equation (6-6) by $e^{iut}$ gives

$$H_{\nu}^{(2)}(\beta, r) e^{iut} \sim \frac{2}{\pi \beta, r} e^{i(\mp \beta, r)} e^{\left(\frac{\pi \nu \pi}{2}\right)}$$

(6-7)

Equation (6-7) corresponds to outgoing waves that decay as $r^{-\nu/2}$. However, since the Hankel function argument is real, energy leaks radially into the soil resulting in the attenuation of the wave propagating along the axis of the pile. This indicates that the wave number, $\xi$, must be complex and contradicts the assumption that the wave number is real. Thus, when the shear wave velocity in the pile is higher than the shear wave velocity in the soil, there are no free modes and only leaky ones are expected.

### 6.2.2 Shear Wave Velocity in the Pile Less than Shear Wave Velocity in the Soil

Suppose that the relative magnitudes of the shear and longitudinal wave velocities in the pile and soil are given as $c_{\nu_p} < c_{\nu_s} < c_{\eta_p} < c_{\eta_s}$. When the phase velocity, $c$, of modes in the pile are lower than the shear and longitudinal wave velocities in the soil, the ratios, $c^2/c_{\nu_p}^2$ and $c^2/c_{\eta_p}^2$ in Equations (6-4) and (6-5) are less than unity and the bracketed terms become negative. Assuming the wave number,
\( \xi \) is real, then \( \alpha_i^2 \) and \( \beta_i^2 \) are also negative and the special function arguments, \( \alpha_r \) and \( \beta_i \) are both purely imaginary.

The displacements in Equations (3-69), (3-70) and (3-71) show that when the special functions are multiplied by the time factor \( e^{i\omega t} \), the appropriate function to satisfy the condition of outgoing travelling waves in the soil is the Modified Bessel’s function, \( K_n \). The asymptotic expression for \( K_n \) for large arguments is given by

\[
K_n(x) \sim \frac{\pi}{\sqrt{2x}} e^{-x}
\]  

(6-8)

Replacing \( x \) by \( i\beta \) and multiplying Equation (6-8) by \( e^{i\omega t} \) gives

\[
K_n(i\beta) e^{i\omega t} \sim \frac{\pi}{\sqrt{2i\beta}} e^{i(\omega t - \beta r)}
\]  

(6-9)

The resulting asymptotic expression in Equation (6-9) corresponds to outgoing waves that decay as \( r^{-1/2} \). The imaginary Modified Bessel function arguments ensure that energy does not leak out into the surrounding soil while satisfying the condition for outward propagating waves that decay in the soil. Therefore, it is possible to obtain free modes with real wave numbers when \( c_r < c_T \). Unfortunately, because piles are
usually made of concrete or steel, the shear wave velocities in piles are typically higher than the shear wave velocities in soils. Consequently, this case is not applicable to the situation of deep foundations.

6.3 ATTENUATION

In the case of free standing piles, i.e. columns, the solution to the frequency equation leads to real branches of the dispersion curves. The wave number is real and the guided wave modes are free or non-attenuating modes. It was shown in Section 6.2 that complex wave numbers are obtained in the solutions of the frequency equation for embedded piles. The complex wave numbers indicate that there are no free modes or real branches in the frequency-wave number domain. The real part of the wave number of the embedded pile is the same as the wave number of the free pile. However, there is an additional imaginary component in the case of the embedded pile. The magnitude of the imaginary component depends on the shear modulus ratio and density ratio between the pile and soil as shown in Section 5.3.2. It will be shown in the following paragraphs that the imaginary component of the wave number is related to attenuation of wave energy. The attenuation described in this dissertation refers to the leakage of energy from the pile into the surrounding soil by geometric damping and does not include material damping.
The loss of energy that causes the wave to attenuate may be quantified by considering the displacements. As shown in Equation (3-132), the displacements in the pile may also be represented as

$$u = u(r) e^{i(\omega - \xi z)}$$  

(6-10)

The wave number is complex and is replaced by $\xi = \xi_r + i\xi_i$, which when substituted into Equation (6-10) gives

$$u = u(r) e^{i(\omega - \xi_r z)} e^{i\xi_i z}$$  

(6-11)

Thus, attenuation of the stress wave propagating along the axis of the pile may be measured by a change in signal amplitude at two arbitrary points along the $z$ direction. The first term in Equation (6-11) is a function of the radius only. By considering a point at some fixed distance along the radius, this term will be constant. The second term represents a wave travelling in the positive $z$ direction with phase velocity, $c = \omega / \xi_r$. The third term corresponds to either an exponentially increasing or an exponentially decreasing quantity depending on the sign of $\xi_i$. The numerical solutions to the frequency equation presented in Chapter 5 show that $\xi_i$ is negative and thus, the displacements decay exponentially in the $z$ direction.
The attenuation of the displacements in the z direction thus depends on the magnitude of the imaginary part of the wave number. The attenuation coefficient, \( \mathcal{G} \) may be expressed in nepers as

\[
\mathcal{G} = \log_e \left( \frac{u_z}{u_i} \right) \text{ nepers}
\]  

(6-12)

where \( u_i \) and \( u_z \) are the amplitude of the signals at the initial and final measuring positions, respectively and \( (u_z/u_i) \) is the amplitude ratio. In Equation (6-11), because the first two terms do not affect the amplitude of the displacements in the z direction, they may be represented by a constant, E. Thus, the attenuation coefficient is expressed as

\[
\mathcal{G} = \log_e \left( \frac{E e^{\xi z_i}}{E e^{\xi z_i}} \right) \\
= \log_e e^{\xi(z_2-z_1)} \\
= \xi(z_2-z_1) \text{ nepers}
\]  

(6-13)

where \( z_i \) and \( z_2 \) are the initial and final measuring positions. The attenuation coefficient expressed in nepers per unit length is then given by

\[
\mathcal{G} = \xi \frac{\text{nepers}}{\text{length}}
\]  

(6-14)
The value of $\xi$, is determined from the imaginary part of the non-dimensional wave number, $\xi a$, by expressing the radius, $a$, in any consistent set of units. Thus, the imaginary part of the wave number corresponds to the attenuation coefficient expressed in terms of nepers per length. The attenuation coefficient expressed in dB per length is given by

$$\mathcal{G} = 8.686 \xi \text{ dB/length}$$ (6-15)

An example of how the attenuation of displacement amplitudes along the length of a pile is determined from the dispersion curves is discussed in the following paragraphs. SI units are used in this example and $a$ is expressed in meters. Modes $1S1$ and $1S2$ were selected on the L(0,1) branch for soft/loose soil while modes $1H1$ and $1H2$ were selected on the L(0,1) branch for hard/dense soil. The modes are shown in Figure 6-1. The attenuation in decibels is calculated for a one-meter diameter concrete pile with lengths ranging 10-30 meters and is summarized in Table 6-1.

The correlation between attenuation in dB and amplitude ratios is given in Table 6-2. The attenuation in dB of -10, -20, -40 and -60, thus corresponds to amplitude ratios of 0.316, 0.1, 0.01 and 0.001, respectively. Therefore, an attenuation of -40 dB to -60 dB results in a final signal where the amplitude has been reduced to 0.1-1.0 % of the original value. Depending on the size of the input signal and the
Figure 6-1  Location of Four L(0,1) Guided Wave Modes in an Embedded Pile on the $\Omega - \xi, a$ Plane
<table>
<thead>
<tr>
<th>Soil Type</th>
<th>Mode No.</th>
<th>Non-Dimensional Frequency $\Omega$</th>
<th>Frequency $f$ (Hz)</th>
<th>Non-Dimensional Wave Number $\xi a$</th>
<th>Attenuation $\xi_i$ (nepers/m)</th>
<th>Attenuation $\xi_i$ (dB)</th>
<th>Length =10m</th>
<th>Length =20m</th>
<th>Length =30m</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft/Loose</td>
<td>1S1</td>
<td>1</td>
<td>732</td>
<td>0.653-0.033</td>
<td>-0.066</td>
<td>-0.573</td>
<td>-6</td>
<td>-11</td>
<td>-17</td>
</tr>
<tr>
<td></td>
<td>1S2</td>
<td>4</td>
<td>2928</td>
<td>4.091-0.168</td>
<td>-0.336</td>
<td>-2.918</td>
<td>-29</td>
<td>-58</td>
<td>-88</td>
</tr>
<tr>
<td>Hard/Dense</td>
<td>1H1</td>
<td>1</td>
<td>732</td>
<td>0.650-0.087</td>
<td>-0.174</td>
<td>-1.511</td>
<td>-15</td>
<td>-30</td>
<td>-45</td>
</tr>
<tr>
<td></td>
<td>1H2</td>
<td>4</td>
<td>2928</td>
<td>4.056-0.446</td>
<td>-0.892</td>
<td>-7.748</td>
<td>-77</td>
<td>-155</td>
<td>-232</td>
</tr>
</tbody>
</table>
Table 6-2  Correlation between Attenuation in dB and Amplitude Ratio

<table>
<thead>
<tr>
<th>Attenuation (dB)</th>
<th>Amplitude Ratio</th>
<th>Amplitude Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>0.89125</td>
<td>89.13%</td>
</tr>
<tr>
<td>-2</td>
<td>0.79433</td>
<td>79.43%</td>
</tr>
<tr>
<td>-3</td>
<td>0.70795</td>
<td>70.79%</td>
</tr>
<tr>
<td>-4</td>
<td>0.63096</td>
<td>63.10%</td>
</tr>
<tr>
<td>-5</td>
<td>0.56234</td>
<td>56.23%</td>
</tr>
<tr>
<td>-6</td>
<td>0.50119</td>
<td>50.12%</td>
</tr>
<tr>
<td>-7</td>
<td>0.44668</td>
<td>44.67%</td>
</tr>
<tr>
<td>-8</td>
<td>0.39811</td>
<td>39.81%</td>
</tr>
<tr>
<td>-9</td>
<td>0.35481</td>
<td>35.48%</td>
</tr>
<tr>
<td>-10</td>
<td>0.31623</td>
<td>31.62%</td>
</tr>
<tr>
<td>-11</td>
<td>0.28184</td>
<td>28.18%</td>
</tr>
<tr>
<td>-12</td>
<td>0.25119</td>
<td>25.12%</td>
</tr>
<tr>
<td>-13</td>
<td>0.22387</td>
<td>22.39%</td>
</tr>
<tr>
<td>-14</td>
<td>0.19953</td>
<td>19.95%</td>
</tr>
<tr>
<td>-15</td>
<td>0.17783</td>
<td>17.78%</td>
</tr>
<tr>
<td>-16</td>
<td>0.15849</td>
<td>15.85%</td>
</tr>
<tr>
<td>-17</td>
<td>0.14125</td>
<td>14.13%</td>
</tr>
<tr>
<td>-18</td>
<td>0.12589</td>
<td>12.59%</td>
</tr>
<tr>
<td>-19</td>
<td>0.11220</td>
<td>11.22%</td>
</tr>
<tr>
<td>-20</td>
<td>0.10000</td>
<td>10.00%</td>
</tr>
<tr>
<td>-25</td>
<td>0.05623</td>
<td>5.62%</td>
</tr>
<tr>
<td>-30</td>
<td>0.03162</td>
<td>3.16%</td>
</tr>
<tr>
<td>-35</td>
<td>0.01778</td>
<td>1.78%</td>
</tr>
<tr>
<td>-40</td>
<td>0.01000</td>
<td>1.00%</td>
</tr>
<tr>
<td>-45</td>
<td>0.00562</td>
<td>0.56%</td>
</tr>
<tr>
<td>-50</td>
<td>0.00316</td>
<td>0.32%</td>
</tr>
<tr>
<td>-55</td>
<td>0.00178</td>
<td>0.18%</td>
</tr>
<tr>
<td>-60</td>
<td>0.00100</td>
<td>0.10%</td>
</tr>
<tr>
<td>-65</td>
<td>0.00056</td>
<td>0.06%</td>
</tr>
<tr>
<td>-70</td>
<td>0.00032</td>
<td>0.032%</td>
</tr>
<tr>
<td>-75</td>
<td>0.00018</td>
<td>0.018%</td>
</tr>
<tr>
<td>-80</td>
<td>0.00010</td>
<td>0.010%</td>
</tr>
<tr>
<td>-85</td>
<td>0.00006</td>
<td>0.006%</td>
</tr>
<tr>
<td>-90</td>
<td>0.00003</td>
<td>0.003%</td>
</tr>
<tr>
<td>-95</td>
<td>0.00002</td>
<td>0.002%</td>
</tr>
<tr>
<td>-100</td>
<td>0.00001</td>
<td>0.001%</td>
</tr>
</tbody>
</table>
signal to noise ratio, \(-40\) dB may considered as the limit after which signals are not discernible.

Thus, from Table 6-1, modes 1S1 and 1H1, which have smaller imaginary part of the wave number (smaller attenuation coefficients), have the ability to propagate larger distances than the corresponding modes 1S2 and 1H2, which have larger imaginary part of the wave number (larger attenuation coefficients). In addition, within the same branch, mode 1S1 will propagate further than mode 1S2. These data also theoretically show why, in impulse response tests, the L(0,1) modes propagate for significant distances only at low frequencies, i.e., for \(\Omega\) less than 2. For a 1 meter diameter concrete pile, this frequency corresponds to 1500 Hz.

### 6.4 PHASE AND GROUP VELOCITIES

The phase and group velocities are calculated from the dispersion curves. As shown in Section 4.7, the phase velocity of a guided wave mode is given by the slope of the secant (line from the origin to the point on the branch represented by the mode) in the \(\Omega - \xi, a\) plot. In this non-dimensional plot, the slope of the secant represents the phase velocity normalized by the shear wave velocity in the pile. Similarly, in the non-dimensional, \(\Omega - \xi, a\) plot, the slope of the tangent at a particular mode represents the group velocity normalized by the shear wave velocity in the pile. The group velocity, which is also known as the velocity of energy flux, represents the propagation velocity
of the group or 'packet' of waves. When using guided waves, the group velocity is used instead of the phase velocity to determine the distances to reflections.

6.4.1 Phase Velocity

The variation of non-dimensional phase velocity with non-dimensional frequency for five longitudinal branches is shown in Figure 6-2. These velocities are essentially independent of the stiffness of the surrounding soil. All the branches exhibit the same general trend except the L(0,1) branch. The L(0,2) and higher branches begin with a steeply decreasing phase velocity which is constant for a small range of Ω, and then gradually becomes asymptotic to the shear wave velocity in the pile at high frequencies. The L(0,1) branch starts at the plateau and decreases steeply to the Raleigh velocity in the pile.

Figure 6-2 shows that the phase velocity remains constant in the narrow range of frequencies represented by the plateaus of each branch. Incidentally, for the L(0,2) and higher branches, the phase velocity in the plateau stage corresponds to the longitudinal wave velocity in the pile. In contrast, the plateau stage for the L(0,1) branch occurs at very low frequencies. It is evident from Figure 6-2 that the phase velocity is not a constant but depends on the frequency or wave number. The assumption that the phase velocity is constant is only valid in certain frequency ranges, and generally, at higher frequencies.
Figure 6-2  Variation of Phase Velocity with Frequency for First Five Branches of Longitudinal Modes in an Embedded Pile
6.4.2 Group Velocity

A simple code was written in Mathematica to numerically interpolate the points on each longitudinal branch in Figure 5-2 using a cubic spline. The Mathematica code generates an expression that can be used to re-plot the curve. The expression is differentiated to provide the derivative, which is then used to determine the group velocity at any point along the curve. These values were also checked with group velocities obtained by taking the slope of the line through points lying immediately on adjacent sides of the selected point. It was found that these two plots of group velocity against frequency were identical.

The variation of group velocity with frequency for five longitudinal branches is shown in Figure 6-3. As in the case of phase velocity, the group velocity is essentially independent of the stiffness of the surrounding soil. The group velocity curves show that the group velocity peaks, falls steeply to a minimum value, and then gradually increases and becomes asymptotic to the Rayleigh velocity in the case of the L(0,1) branch, and the shear wave velocity in the case of the L(0,2) and higher branches. The peak group velocity for the L(0,2) and higher branches is one and half times the shear wave velocity in the pile while the L(0,1) branch shows a slightly higher peak. The higher branches exhibit more than one peak and the peaks line up with the peaks of the adjacent lower order branch.
Figure 6-3  Variation of Group Velocity with Frequency for First Five Branches of Longitudinal Modes in an Embedded Pile
At very low frequencies, the non-dimensional group velocity of the L(0,1) mode is approximately 1.54. The ratio of the bar velocity to the shear wave velocity in the pile is 1.52. This explains the common assumption used in the Sonic Echo and Impact Response tests where the wave speed is usually taken as the bar wave velocity of the concrete pile. This assumption is valid as long as low frequencies are used in these tests.

6.5 FREQUENCY RANGE FOR CONCRETE PILES

In the sonic echo and impulse response tests, an impact hammer, which is used to strike the pile head, is designed to generate stress waves with frequencies in the range of 0-2 kHz. Special aluminum tipped hammers can input frequencies as high as 5 kHz. However, high frequencies are not used in testing deep foundations as the assumption that the wave propagation is one-dimensional requires that wavelength be larger than the pile diameter. Hence, these tests are limited to the low frequency range. The accuracy of these assumptions for the L(0,1) mode have also been discussed in Section 6.4.

The frequency range over which the guided wave approach is applicable in the non-destructive evaluation of deep concrete foundations is now investigated. The main concern is whether high frequency modes can propagate in an embedded pile and whether there is an upper bound limit for frequency. It is recognized for waves
propagating in a concrete pile that energy losses become dominant due to scattering at wavelengths smaller than the aggregate size. Thus, for the lower bound limit on the wavelength, it would be sufficient to consider wavelengths that are larger than the maximum aggregate size typically used for the construction of drilled shafts. There is no upper bound limit on the wavelength.

The wavelength, $\lambda$, is related to the wave number by

$$
\xi = \frac{2\pi}{\lambda}
$$

(6-16)

The frequency is related to the angular frequency by

$$
\omega = 2\pi f
$$

(6-17)

Substituting Equations (6-16) and (6-17) into (6-3) gives

$$
f = \frac{c}{\lambda}
$$

(6-18)

Thus, the frequency depends on the phase velocity and the wavelength. The phase velocity is not a constant as evident from the results shown in Figure 6-2. The phase velocity is shown to vary drastically in the low frequencies but is asymptotic to
the shear wave velocity in the pile in the high frequency range. For the purposes of determining the bounds for frequency, the phase velocity is taken as the shear wave velocity in the pile. This is a good approximation in the high frequency range.

Approximating the shear wave velocity in the drilled shaft as 2300 m/s, the maximum theoretical angular frequency for guided wave propagation can be calculated using Equation (6-18). Table 6-3 shows the maximum angular and non-dimensional frequencies for drilled shafts with diameters ranging between 0.5-3 meters (1.6-9.8 ft.) and typical values of aggregate sizes of 38-76 mm (1.5-3 in.). It is evident that the size of the aggregates used in the drilled shaft determines the maximum angular frequency that can be used in guided wave propagation. For a typical drilled shaft diameter of 1 meter and aggregate size of 2 inches, angular frequencies above 45 kHz or non-dimensional frequencies above 62 may be ignored due to high-energy losses.

6.6 POWER AND DISPLACEMENT DISTRIBUTION

In addition to the attenuation described in Section 6.3, the power and displacement profiles of guided wave modes are important considerations in evaluating modes that can be induced in a deep foundation and propagate significantly in the pile. In Section 4.6, the power or energy flux in a guided wave mode, flowing axially through a cross-section of the pile, was normalized so that displacements or
Table 6-3 Maximum Frequencies for Concrete Piles

<table>
<thead>
<tr>
<th>Maximum Aggregate Size (mm)</th>
<th>Pile Diameter (m)</th>
<th>Maximum Frequency $f_{max}$ (kHz)</th>
<th>Max. Non-Dimensional Frequency ($\Omega_{max}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>38 (1.5 in)</td>
<td>0.5</td>
<td>61</td>
<td>41</td>
</tr>
<tr>
<td>38</td>
<td>1.0</td>
<td>61</td>
<td>83</td>
</tr>
<tr>
<td>38</td>
<td>2.0</td>
<td>61</td>
<td>165</td>
</tr>
<tr>
<td>38</td>
<td>3.0</td>
<td>61</td>
<td>248</td>
</tr>
<tr>
<td>51 (2.0 in)</td>
<td>0.5</td>
<td>45</td>
<td>31</td>
</tr>
<tr>
<td>51</td>
<td>1.0</td>
<td>45</td>
<td>62</td>
</tr>
<tr>
<td>51</td>
<td>2.0</td>
<td>45</td>
<td>123</td>
</tr>
<tr>
<td>51</td>
<td>3.0</td>
<td>45</td>
<td>185</td>
</tr>
<tr>
<td>76 (3.0 in)</td>
<td>0.5</td>
<td>30</td>
<td>21</td>
</tr>
<tr>
<td>76</td>
<td>1.0</td>
<td>30</td>
<td>41</td>
</tr>
<tr>
<td>76</td>
<td>2.0</td>
<td>30</td>
<td>83</td>
</tr>
<tr>
<td>76</td>
<td>3.0</td>
<td>30</td>
<td>124</td>
</tr>
</tbody>
</table>

stresses from any mode could be compared directly. The axial power in the mode is expressed in terms of two radial components, a shear component given by $\sigma_r\,u_r^*$ and a normal component given by $\sigma_r\,u_r^*$. These radial components of power relate to the modal energy leaking radially into the soil.

The expressions for the displacements in the pile were derived in Section 4.4. The axial and radial displacements are complex quantities, and in the theoretical formulation, the real part is taken as the solution. However, due to the harmonic nature
of the expressions, depending on the phase of the wave, either the real part represents the displacements and some multiple of the imaginary part represents the velocities or the imaginary part represents the displacements and some multiple of the real part represents the velocity. The real and imaginary components of the axial and radial displacements of the modes are plotted to show the displacement and velocity profiles, which indicate the shape which the modes should be induced.

Seventeen modes were selected based on their low attenuation coefficients, and at locations of maximum and minimum group velocities. Figure 6-4 shows the locations of the modes on the $\Omega - \xi, a$ plot and Figure 6-5 shows their locations on the non-dimensional group velocity versus non-dimensional frequency plot. The modes are named according to the branch they belong to, followed by a letter to differentiate the location on that particular branch. The frequencies and wave numbers of the selected guided wave modes are given in Table 6-4. The power and displacement profiles of the modes are included in the Appendix.

From a purely attenuation perspective, any mode that has lower attenuation compared to the L(0,1)a or L(0,1)b modes, theoretically, propagates greater distances than the one-dimensional modes. The issue would then be whether it would be possible to induce the mode in practice. Thus, the combination of low attenuation and practical means of inducing the guided wave mode are important considerations in selecting modes for propagation in deep foundations.
Figure 6-4  Location of Selected Guided Wave Modes in an Embedded Pile on the $\Omega - \xi, a$ Plane
Figure 6-5  Location of Selected Guided Wave Modes on Group Velocity-Frequency Curves
Table 6-4  Frequencies and Wave Numbers of Selected Guided Wave Modes

<table>
<thead>
<tr>
<th>Mode</th>
<th>Non-Dimensional Frequency (Ω)</th>
<th>Real part of Non-Dimensional Wave Number (ξ,α)</th>
<th>Imaginary part of Non-Dimensional Wave Number (ξ,α)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(0,1)a</td>
<td>0.1</td>
<td>0.065</td>
<td>-0.034</td>
</tr>
<tr>
<td>L(0,1)b</td>
<td>0.937</td>
<td>0.611</td>
<td>-0.033</td>
</tr>
<tr>
<td>L(0,1)c</td>
<td>2.978</td>
<td>2.39</td>
<td>-0.165</td>
</tr>
<tr>
<td>L(0,1)d</td>
<td>10.026</td>
<td>11.121</td>
<td>-0.275</td>
</tr>
<tr>
<td>L(0,2)a</td>
<td>5.065</td>
<td>3.17</td>
<td>-0.049</td>
</tr>
<tr>
<td>L(0,2)b</td>
<td>7.825</td>
<td>5.726</td>
<td>-0.066</td>
</tr>
<tr>
<td>L(0,2)c</td>
<td>12.339</td>
<td>11.331</td>
<td>-0.032</td>
</tr>
<tr>
<td>L(0,2)d</td>
<td>26.994</td>
<td>26.648</td>
<td>-0.007</td>
</tr>
<tr>
<td>L(0,3)a</td>
<td>8.985</td>
<td>5.61</td>
<td>-0.049</td>
</tr>
<tr>
<td>L(0,3)b</td>
<td>12.045</td>
<td>8.499</td>
<td>-0.067</td>
</tr>
<tr>
<td>L(0,3)c</td>
<td>19.412</td>
<td>17.638</td>
<td>-0.033</td>
</tr>
<tr>
<td>L(0,4)a</td>
<td>12.977</td>
<td>8.1</td>
<td>-0.049</td>
</tr>
<tr>
<td>L(0,4)b</td>
<td>16.534</td>
<td>11.677</td>
<td>-0.067</td>
</tr>
<tr>
<td>L(0,4)c</td>
<td>26.646</td>
<td>24.13</td>
<td>-0.032</td>
</tr>
<tr>
<td>L(0,5)a</td>
<td>17.0</td>
<td>10.611</td>
<td>-0.048</td>
</tr>
<tr>
<td>L(0,5)b</td>
<td>20.5</td>
<td>14.129</td>
<td>-0.071</td>
</tr>
<tr>
<td>L(0,5)c</td>
<td>33.3</td>
<td>29.988</td>
<td>-0.033</td>
</tr>
</tbody>
</table>
6.6.1 Comparison of L(0,1)a, L(0,1)b, L(0,1)c and L(0,1)d Modes

These four modes are located on the L(0,1) branch as shown in Figure 6-4. The L(0,1)a and L(0,1)b modes are located on the portion of the curve where the frequencies are low and the attenuation coefficients are small and approximately constant. The L(0,1)c and L(0,1)d modes have increasingly higher frequencies and larger attenuation coefficients. The power and displacement plots of all the four modes are shown in Figure 6-6 and Figure 6-7, respectively.

The normalized axial power in the L(0,1)a mode varies linearly from zero at the center of the pile (R=0) to unity at the pile-soil interface (R=1). The real and imaginary components of the radial displacements are almost zero throughout the cross-section. However, the real and imaginary components of the axial displacements are constant along the radius, which agrees with the linear trend of the axial power. This behavior is consistent with the assumption that for long wavelengths, one-dimensional wave propagation can be assumed, where the waves travelling down the pile are considered as plane waves. It would be expected that in such cases, there would only be axial displacements and no radial displacements, which is validated by the displacements profiles. The uniform axial displacement profile over the cross-section of the pile indicates that in practice, it would be relatively easy to induce the L(0,1)a mode in an embedded pile.
Figure 6-6  Normalized Axial Power Distributions for Modes on the L(0,1) Branch
Figure 6-7  Normalized Displacement Distributions for Modes on the L(0,1) Branch
The axial power in the L(0,1)b mode shown in Figure 6-6 varies linearly while the axial displacement components are nearly constant over the cross-section. The radial displacement components are small compared to the axial displacements. This mode is similar to the L(0,1)a mode. For the L(0,1)c and L(0,1)d modes, the axial power is non-linear as shown in Figure 6-6. The power in the L(0,1)d mode is concentrated near the pile-soil interface and the axial and radial displacements are confined to this region, indicating that the leakage of energy into the surrounding soil is greater for this mode. The higher frequency modes have extremely high attenuation coefficients and will decay rapidly compared to the lower frequency modes. Thus, the L(0,1)c and L(0,1)d modes will not propagate large distances and can be excluded.

6.6.2 Comparison of L(0,2)a, L(0,3)a, L(0,4)a and L(0,5)a Modes

These modes were selected as modes having peak group velocities in their respective branches as shown in Figure 6-5. Incidentally, the imaginary part of the non-dimensional wave number of these modes is the same and when plotted in Figure 6-4, the four modes line up vertically. These modes propagate at the maximum group velocities and may be isolated from the slower adjacent modes. The displacement profiles of the modes are given in Figure 6-8. The axial and radial displacements in these modes oscillate and the oscillations increase with increasing order of the branch. Due to the oscillatory nature of the axial displacement profiles of the higher order modes, it would be difficult to excite these complex mode shapes in an embedded pile.
Figure 6-8 Normalized Displacement Distributions for Modes with Maximum Group Velocities
Thus, only the L(0,2)a and L(0,3)a modes which have simpler axial displacement profiles and the ability to propagate at peak group velocities are likely suitable for propagation in a deep foundation.

6.6.3 Comparison of L(0,2)b, L(0,3)b, L(0,4)b and L(0,5)b Modes

These modes were selected as modes having the minimum group velocities in their respective branches as shown in Figure 6-5. The imaginary part of the non-dimensional wave number for these four modes is coincidentally the same. These modes line up vertically in Figure 6-4, the same phenomenon observed in Section 6.6.2 for modes with peak group velocities. The displacement profiles of the four modes are given in Figure 6-9. The axial and radial displacements in these modes oscillate and the oscillations increase with increasing order of the branch. The oscillations in these modes have more peaks and troughs compared to the modes with maximum group velocities which were described in Section 6.6.2. Thus, the more complicated modal shapes coupled with the higher attenuation coefficients of this series of modes make them less suitable for propagation than the modes with peak group velocities.

6.6.4 Comparison of L(0,2)c, L(0,3)c, L(0,4)c and L(0,5)c Modes

These modes were selected based on modes having the same imaginary component of the non-dimensional wave number as the L(0,1)b mode. The
Figure 6-9  Normalized Displacement Distributions for Modes with Minimum Group Velocities
displacement profiles of these four modes are given in Figure 6-10. The axial and radial displacements are oscillatory and the oscillations increase with increasing order of the branch. The axial displacement profiles of the higher order modes have complex modal shapes and it would be difficult to excite these modes in an embedded pile. Thus, only the L(0,2)c and L(0,3)c modes which have simpler axial displacement profiles may be suitable for propagation.

6.6.5 Comparison of L(0,2)a, L(0,2)b, L(0,2)c and L(0,2)d Modes

These four modes are located on the L(0,2) branch as shown in Figure 6-4. The L(0,2)a, L(0,2)b and L(0,2)c modes were considered in the previous sections. The L(0,2)d mode has the highest frequency and the smallest attenuation coefficient in this series. The displacement plots of all the four modes are shown in Figure 6-11. The oscillations in the axial and radial displacements are few and the modal shape is generally parabolic in nature. The uncomplicated modal shape of the L(0,2) modes make modes with small attenuation coefficients on this branch suitable for propagation in an embedded pile.

6.7 PRACTICAL IMPLICATIONS

The two main criteria that constitute the characteristics of an ideal mode for propagation are the attenuation of the mode and the ability to induce the mode in the
Figure 6-10  Normalized Displacement Distributions for Modes with Same Attenuation Coefficient as the L(0,1)b Mode
Figure 6-11  Normalized Displacement Distributions for Modes on the L(0,2) Branch
pile. The attenuation was discussed in Section 6.3, and the ability to induce the mode, which is related to the displacement profiles, was discussed in the previous section.

In the very low frequency range where $\Omega < 2$, the $L(0,1)$ modes exhibit uniform displacement profiles that are relatively easy to induce in the pile. The modes have constant imaginary part of the wave number in this range and thus, the attenuation is constant. As frequency increases, the attenuation of this mode increases rapidly and the $L(0,1)$ modes do not propagate.

From an attenuation standpoint, modes on the $L(0,2)$ branch show great promise as the attenuation of these modes are the lowest compared to all other branches for non-dimensional frequencies above 12. The displacement profiles of the $L(0,2)$ show less than a single oscillation and these modes can potentially be induced in a pile. The modes from the $L(0,3)$ and higher branches, show increasing oscillations that increase further with frequency, and therefore make it more difficult to induce them even though the attenuation decreases at higher frequencies.

6.8 SUMMARY

In non-destructive evaluation of deep foundations, the material constants of the pile and soil show that only leaky guided wave modes are obtained. The modes, which are similar to that obtained from the free pile case, propagate along the pile but decay.
The decay is represented by the imaginary part of the wave number. The attenuation of the waves can be represented directly by the imaginary part of the wave number in nepers per length. A smaller imaginary component leads to a lower attenuation, which results in larger propagation distances in a pile.

The phase and group velocities were shown to vary with the frequency or wave number, contrary to the assumptions made in the one-dimensional approach. The phase and group velocities are asymptotic to the Rayleigh wave velocity in the pile for the L(0,1) branch and the shear wave velocity in the pile for rest of the higher order longitudinal branches.

Seventeen modes were selected for assessment based on low attenuation coefficients, as well as maximum and minimum group velocities. The mode shapes of the guided wave modes were evaluated by comparing their axial and radial displacement profiles. For non-dimensional frequencies less than 2, the L(0,1) modes have the lowest attenuation and are easily induced in a pile. Modes on the L(0,2) branch are likely to be induced in practice and have the least attenuation compared to all other modes for non-dimensional frequencies above 12. The modes on the L(0,2) and L(0,3) branches with peak group velocities are isolated, which make it feasible for these modes to be propagated in practice.
Chapter 7

Summary and Conclusions

7.1 SUMMARY

A theoretical approach to non-destructive evaluation of deep foundations using guided waves was presented. The guided wave approach, based on three-dimensional wave propagation, provides a theoretical improvement over the one-dimensional wave propagation approach used in impact/sonic echo and impulse response tests, popular non-destructive evaluation tests for deep foundations. These surface reflection tests were described and the assumptions and limitations in the methods were discussed. The results of two field investigations performed to evaluate the capabilities of the non-destructive tests to predict defects in deep foundations were presented. Previous work on guided wave propagation in plates, rods and hollow cylinders was outlined.

Guided wave propagation in an infinitely long cylindrical pile embedded in soil was developed from the basic equations of elasticity. Appropriate potential functions were selected to account for harmonic waves propagating axially along the pile and radially outwards into the soil. The general expressions for displacements and stresses in the pile and soil were derived. For the pile and soil to be fully bonded at the pile-soil interface, the lateral boundary conditions stipulate the continuity of displacements and stresses at the interface. Applying the boundary conditions resulted in a system of
homogeneous equations. The homogeneous system of equations has non-trivial solutions only when the determinant vanishes. Thus, by setting the determinant to zero, the general frequency equation for an embedded pile was obtained. The general frequency equation decomposes into products of simpler sub-determinants when specific motions are considered. The frequency equations for longitudinal and torsional modes were obtained by considering axisymmetric motion and only longitudinal modes were considered in this study.

The frequency equation for longitudinal modes was presented in a non-dimensional form by a $4 \times 4$ determinant. Non-dimensional variables, which constituted the components of the determinant, were defined in terms of the Poisson's ratios, shear moduli and densities of the pile and soil. The frequency equation represents a transcendental relationship between the non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$. The axial and radial displacements and stresses in the pile and soil were derived for the case of longitudinal modes. A normalizing factor, which sets the energy flux or power of any mode through a cross-section of the pile to unity, was derived so that displacements or stresses of any mode could be compared directly. The phase and group velocities of guided wave modes were defined, and these velocities were computed from the dispersion curves.

A numerical code was developed using a symbolic mathematical package to plot the dispersion curves for longitudinal modes in an embedded pile. The longitudinal
modes were represented by an infinite number of branches in the frequency-complex wave number domain. The first five branches of longitudinal modes were plotted for a practical range of soil conditions that varied from soft/loose soils to hard/dense soils. The soft/loose and hard/dense soils were represented by shear modulus ratios of $\mu_p/\mu_s = 325$ and $\mu_p/\mu_s = 45$, respectively.

The theoretical formulation of the frequency equation for the embedded pile was verified analytically by showing that the 4x4 determinant in the frequency equation was reduced to the Pochhammer-Chree frequency equation for a free rod as the soil surrounding the pile became “air-like”. The computer code was verified numerically when the first five branches of longitudinal modes in a free pile, computed using the embedded pile code were in excellent agreement with the corresponding branches obtained from the free pile code. The behavior of longitudinal modes in an embedded pile was studied by examining the effect of shear modulus and density ratios on the transition from the free pile to the embedded pile.

The material constants of the pile and soil were investigated to check whether propagating modes could be obtained in the case of embedded concrete piles in soils. The modes, which are similar to that obtained from the free pile case, propagate along the pile but decay with distance. The decay is represented by the imaginary part of the wave number. The attenuation of the waves, due to geometric damping, can be
represented directly by the imaginary part of the wave number in nepers per length. The phase and group velocities were shown to vary with the frequency or wave number, contrary to the assumptions made in the one-dimensional approach. The phase and group velocities are asymptotic to the Rayleigh wave velocity in the pile for the $L(0,1)$ branch and to the shear wave velocity in the pile for rest of the higher order longitudinal branches.

The combination of low attenuation and practical means of inducing the guided wave mode are important considerations in selecting modes for propagation in deep foundations. For non-dimensional frequencies less than 2, the $L(0,1)$ modes have the lowest attenuation and are easily induced in a pile. Modes on the $L(0,2)$ branch have relatively simple modal shapes, which suggests that they may be induced in practice, and have the least attenuation compared to all other modes for non-dimensional frequencies above 12. The modes with peak group velocities are isolated, which make it feasible that these particular modes can be induced in practice.

7.2 CONCLUSIONS

Based on the theoretical formulation and numerical analyses of dispersion curves for longitudinal modes in a pile embedded in soil, the following conclusions are made:
(a) Guided wave propagation in an infinitely long cylindrical pile embedded in soil can be represented by a general frequency equation that governs all the various motions in the pile. The general frequency equation decomposes into the frequency equation for longitudinal and torsional modes when the simple case of axisymmetric motion in the pile is considered. The longitudinal modes involve coupled dilatational and equivoluminal displacements.

(b) The frequency equation for longitudinal modes, expressed as a transcendental relationship between the non-dimensional frequency, \( \Omega \), and non-dimensional wave number, \( \xi a \), is defined in terms of the Poisson’s ratios, shear moduli and densities of the pile and soil.

(c) The solution to the frequency equation for longitudinal modes is satisfied by an infinite number of modes that form branches in the \( \Omega - \xi, a - \xi, a \) (non-dimensional frequency versus real part of non-dimensional wave number versus imaginary part of non-dimensional wave number) space. The first five branches of longitudinal modes were numerically evaluated for a concrete pile embedded in soft/loose soils, where \( \mu_p/\mu_s = 325 \), and in hard/dense soils, where \( \mu_p/\mu_s = 45 \). The results showed that the branches for longitudinal modes for both soils were essentially identical in the \( \Omega - \xi, a \) plane, but the hard/dense soil showed a higher imaginary component in the \( \Omega - \xi, a \) plane.
(d) All the branches of longitudinal modes generally showed larger imaginary values at low frequencies and progressively smaller values at higher frequencies except the $L(0,1)$ branch, which showed an opposite trend.

(e) Sensitivity analysis showed that the shear modulus ratio, $\mu_p/\mu_s$, and density ratio, $\rho_p/\rho_s$, had a negligible effect on the real part of the non-dimensional wave number but a significant effect on the imaginary part of the wave number.

(f) The real part of the wave number for the first five branches of longitudinal modes in an embedded concrete pile were essentially identical to the corresponding wave numbers of free modes in a free-standing concrete pile, except at the cut-off frequencies. Real wave numbers correspond to free modes, i.e. modes that are non-attenuating.

(g) The wave numbers of longitudinal modes in the embedded pile were complex because the shear wave velocity in a concrete pile is always higher than the shear wave velocity in soil. Complex wave numbers correspond to leaky modes, i.e. modes that propagate but decay with distance traveled.
(h) The attenuation of the longitudinal guided wave modes, as a result of geometric damping, is represented directly by the imaginary part of the wave number, $\xi$, in nepers per length.

(i) The phase and group velocities vary with the frequency and wave number, contrary to the assumptions made in the theory of one-dimensional approach. The phase and group velocities of the $L(0,1)$ branch are asymptotic to the Rayleigh wave velocity in the pile, while the velocities of the higher order longitudinal branches are asymptotic to the shear wave velocity.

(j) The assumption in the impact/sonic echo and impulse response tests that at low frequencies the waves propagate as plane waves and the phase velocity is constant, is verified. However, this assumption is only valid for non-dimensional frequencies less than 2.

(k) For non-dimensional frequencies less than 2, the $L(0,1)$ modes have the lowest attenuation and are easily induced in a pile as evidenced by the popularity of the impulse response test in practice. For concrete piles with diameters of 0.5-2.0 meters, $\Omega = 2$ corresponds to frequencies of 3000-750 Hz. Because of their relative simple mode shapes, modes on the $L(0,2)$ branch are likely to be induced in practice. They have the least attenuation compared to all other
modes for non-dimensional frequencies above 12. For concrete piles with
diameters of 0.5-2.0 meters, $\Omega=12$ corresponds to frequencies of 17.8-4.4
kHz. The modes with peak group velocities are isolated, which make it feasible
for these modes to be induced in practice.


FREE PILE CODE

Clear[list01, solu, omega, alpha, gg, hh, kk, n, v, eq3];
eq3 = (-2*gg^2 + omega^2)^2*BesselJ[1, kk]*BesselJ[0, hh] -
2*hh*BesselJ[1, hh]*(-omega^2*BesselJ[1, kk] + 2*gg^2*kk*BesselJ[0, kk]);
list01 = {};
v = 0.18;
n = 0.0735809;
alpha = Sqrt[(1 - 2*v)/(2*(1 - v))];
Do[hh = Sqrt[alpha^2*omega^2 - gg^2]; kk = Sqrt[omega^2 - gg^2];
solu = FindRoot[eq3 == 0, {omega, n}];
AppendTo[list01, {gg, solu[[1, 2]]}];
Print[{v, alpha, hh, kk, gg, n, solu[[1, 2]]}]; n = solu[[1, 2]],
{gg, 0.01, 0.08, 0.01}]

EMBEDDED PILE CODE

Frequency-Wave Number Routine

Clear[matrixc,c, U, V, W, X, Y, alpha1, alpha2, wavenumberx, densityratio, shearwavevelocityratio, n1, n2, lamda1, lamda2, mu1, mu2, muratio, nu1, nu2, omegax];
Print[matrixc,c, U, V, W, X, Y, alpha1, alpha2, wavenumberx, densityratio, shearwavevelocityratio, lamda1, lamda2, mu1, mu2, muratio, nu1, nu2, omegax];
matrixc := Array[c, 4, 4];
c[1, 1] := V*BesselJ[1, V];
c[1, 2] := Y*BesselJ[1, X];
c[1, 3] := W*(BesselJ[1, W] - I*BesselY[1, W]);
c[1, 4] := Y*(BesselJ[1, U] - I*BesselY[1, U]);
c[2, 1] := -Y*BesselJ[0, V];
c[2, 2] := X*BesselJ[0, X];
c[2, 3] := -Y*(BesselJ[0, W] - I*BesselY[0, W]);
c[2, 4] := U*(BesselJ[0, U] - I*BesselY[0, U]);
c[3, 1] := ((2*V^2)+(lamda1*(V^2 + Y^2))/mu1)*BesselJ[0, V] - 2*V*BesselJ[1, V];
c[3, 2] := 2*Y*(X*BesselJ[0, X] - BesselJ[1, X]);
c[3, 3] := (BesselJ[0, W]-I*BesselY[0, W])*)((2*mu2*W^2/mu1)+(W^2 + Y^2)*lamda2/mu1)-(2*W*(BesselJ[1, W]-I*BesselY[1, W])*mu2/mu1);
c[3, 4] := (1/mu1)*2*Y*mu2*(U*(BesselJ[0, U] - I*BesselY[0, U])-(BesselJ[1, U] - I*BesselY[1, U]));
c[4, 1] := 2*V*Y*BesselJ[1, V];
c[4, 2] := -(X^2 - Y^2)*BesselJ[1, X];
c[4, 4] := -mu2*(U^2 - Y^2)*(BesselJ[1, U] - I*BesselY[1, U])/mu1;
mu1 := muratio/mu2;
shearwavevelocityratio := Sqrt[muratio/densityratio];
lamda1 := 2*nu1*mu1/(1 - 2*nu1);
lamda2 := 2*nu2*mu2/(1 - 2*nu2);
V := Sqrt[alpha1^2*omegax^2 - Y^2];
X := Sqrt[omegax^2 - Y^2];
W := Sqrt[alpha2^2*shearwavevelocityratio^2*omegax^2 - Y^2];
U := Sqrt[shearwavevelocityratio^2*omegax^2 - Y^2];
Y := wavenumberx;
alpha1 := Sqrt((1 - 2*nu1)/(2*(1 - nu1)));
alpha2 := Sqrt((1 - 2*nu2)/(2*(1 - nu2)));
nu1 := 0.18; nu2 := 0.3;
muratio := 325.; densityratio := 1.2;
EMBEDDED PILE CODE

Null Space Vector Routine

Clear[matrxc,c, U, V, W, X, Y, alpha1, alpha2, wavenumberx, densityratio, shearwavevelocityratio, n1, n2, lambd1, lambd2, mu1, mu2, muratio, nu1, nu2, omegax];
Print[{matrxc,c, U, V, W, X, Y, alpha1, alpha2, wavenumberx, densityratio, shearwavevelocityratio, lambd1, lambd2, mu1, mu2, muratio, nu1, nu2, omegax}]

matrxc := Array[c, {4, 4}];
c[1, 1] := -V*BesselJ[1, V];
c[1, 2] := Y*BesselJ[1, X];
c[1, 3] := W*(BesselJ[1, W] - I*BesselY[1, W]);
c[1, 4] := -Y*(BesselJ[1, U] - I*BesselY[1, U]);
c[2, 1] := -Y*BesselJ[0, V];
c[2, 2] := -X*BesselJ[0, X];
c[2, 3] := Y*(BesselJ[0, W] - I*BesselY[0, W]);
c[2, 4] := U*(BesselJ[0, U] - I*BesselY[0, U]);
c[3, 1] := -(2*V^2)+(lambd1*(V^2 + Y^2))/mu1)*BesselJ[0, V] + 2*V*BesselJ[1, V];
c[3, 2] := 2*Y*(X*BesselJ[0, X] - BesselJ[1, X]);
c[3, 3] := (BesselJ[0, W]-I*BesselY[0, W])*(2*mu2*W^2/mu1)+(W^2 + Y^2)*lambd2/mu1) - (2*W*(BesselJ[1, W] - I*BesselY[1, W])*mu2/mu1);  
c[3, 4] := -(1/mu1)*2*Y*mu2*(U*(BesselJ[0, U] - I*BesselY[0, U]) - (BesselJ[1, U] - I*BesselY[1, U]));
c[4, 1] := 2*V*Y*BesselJ[1, V];
c[4, 2] := (X^2 - Y^2)*BesselJ[1, X];
c[4, 4] := mu2*(Y^2 - U^2)*(BesselJ[1, U] - I*BesselY[1, U])/ mul1;

mul1 := muratio*mu2;
shearwavevelocityratio := Sqrt[muratio/densityratio];
lambd1 := 2*nu1*mu1/(1 - 2*nu1);
lambd2 := 2*nu2*mu2/(1 - 2*nu2);
V := Sqrt[alpha1^2*omegax^2 - Y^2];
X := Sqrt[omegax^2 - Y^2];
W := Sqrt[alpha2^2*shearwavevelocityratio^2*omegax^2 - Y^2];
U := Sqrt[shearwavevelocityratio^2*omegax^2 - Y^2];
Y := wavenumberx;
alpha1 := Sqrt[(1 - 2*nu1)/(2*(1 - nu1))];
alpha2 := Sqrt[(1 - 2*nu2)/(2*(1 - nu2))];
listwave=listwave$325
EMBEDDED PILE CODE

Null Space Vector Routine (cont’d)

nu1 = 0.18;
nu2 = 0.3;
muratio = 325.;
densityratio = 1.2;

Clear[m1,m2,m3];
nullspacelist = { }
Do[omegax = listwave[[i, 1]];
   wavenumberx = listwave[[i, 2]];
   m1 = LUDecomposition[matrixc];
   Print[m1];
   m2 = Flatten[Take[m1, 1, 1];
   m3 = ReplacePart[m2, 0, {{2, 1}, {3, 1}, {3, 2}, {4, 1}, {4, 2}, {4, 3}}];
   Print[MatrixForm[m3]];
   m4 = ReplacePart[m2, 0, {{2, 1}, {3, 1}, {3, 2}, {4, 1}, {4, 2}, {4, 3}, {4, 4}}];
   ns = NullSpace[m4];
   AppendTo[nullspacelist, {omegax, wavenumberx, ns}];
   Print["Frequency = ", omegax, ", Wavenumber = ", wavenumberx, ", ns", {i, 1, 299}]"
EMBEDDED PILE CODE

Displacement and Power Routine

Clear[RadDispln, RadDispOut, AxiDispln, AxiDispOut, matrixc, c, U, V, W, X, Y, 
alpha1, alpha2, wavenumber, densityratio, shearwavevelocityratio, lamda1, lamda2, 
mu1, mu2, muratio, nu1, nu2, omega, a, r, R, ns1, ns2, ns3, ns4, SigmaRR, SigmaRZ, 
SrrUrstar, ImSrrUrstar, SrzUzstar, ImSrzUzstar];

Print[{{RadDispln, RadDispOut, AxiDispln, AxiDispOut, matrixc, c, U, V, W, X, Y, 
alpha1, alpha2, wavenumber, densityratio, shearwavevelocityratio, lamda1, lamda2, 
mu1, mu2, muratio, nu1, nu2, omega, a, r, R, ns1, ns2, ns3, ns4, SigmaRR, SigmaRZ, 
SrrUrstar, ImSrrUrstar, SrzUzstar, ImSrzUzstar}];

mu2=mu1/muratio;
shearwavevelocityratio=Sqrt[muratio/densityratio];
lamda1 = 2*nu1*mu1/(1 - 2*nu1);
lamda2 = 2*nu2*mu2/(1 - 2*nu2);
V = Sqrt[alpha1^2*omega^2 - Y^2];
X = Sqrt[omega^2 - Y^2];
W = Sqrt[alpha2^2*shearwavevelocityratio^2*omega^2 - Y^2];
U = Sqrt[shearwavevelocityratio^2*omega^2 - Y^2];
Y = wavenumber;

alpha1 = Sqrt[(1 - 2*nu1)/(2*(1 - nu1))];
alpha2 = Sqrt[(1 - 2*nu2)/(2*(1 - nu2))];

RadDispln=(-V*BesselJ[1, V*R]*ns1)+(Y*BesselJ[1, X*R]*ns2);
AxiDispln=-I*((Y*BesselJ[0, V*R]*ns1)+(X*BesselJ[0, X*R]*ns2));
SigmaRR=((-lamda1/mu1*(V^2+Y^2)+2*V^2)*BesselJ[0,V*R] +2*V/R*
BesselJ[1,V*R])*ns1 +2*Y*(X*BesselJ[0,X*R]-I/R*BesselJ[1,X*R])*ns2);
SigmaRZ=I*(2*V*Y*BesselJ[1,V*R]*ns1+(X^2-Y^2)*BesselJ[1,X*R]*ns2);
RadDispOut=((W*(BesselJ[1, W*R] - I*BesselY[1, W*R])*ns3)+(Y*(BesselJ[1, 
U*R] - I*BesselY[1, U*R])*ns4));
AxiDispOut=I*((Y*(BesselJ[0, W*R] - I*BesselY[0, W*R])*ns3)+(U*(BesselJ[0, 
U*R] - I*BesselY[0, U*R])*ns4));
<<Graphics'Legend'

nu1=0.18;
nu2 = 0.30;
muratio=325.;
densityratio = 1.2;
EMBEDDED PILE CODE

Displacement and Power Routine (cont’d)

Print["nu pile=",nu1]; Print["nu soil=",nu2];
Print["shear modulus ratio=",muratio];
Print["density ratio=",densityratio];
Print[" "]; Print[" "];
Do[omega=NullSpacelist[i,1]];
wavenumber=NullSpacelist[i,2];
ns1=NullSpacelist[i,3,1,1];
ns2=NullSpacelist[i,3,1,2];
ns3=NullSpacelist[i,3,1,3];
ns4=NullSpacelist[i,3,1,4];

RDI=RadDispIn/.R->1;
RDO=RadDispOut/.R->1;
ADI=AxiDispIn/.R->1;
ADO=AxiDispOut/.R->1;

SrrUrstar=SigmaRR*Conjugate[RadDispIn];
ImSrrUrstar=Im[SrrUrstar];
SrzuUrstar=SigmaRZ*Conjugate[AxiDispIn];
ImSrzuUrstar=Im[MrzuUrstar];

C=Sqrt[(2*Im[Y])/(Pi*omega*(ImSrrUrstar+ImSrzuUrstar))]/.R->1;
C^2*Pi*omega/(2*Im[Y])*(ImSrrUrstar+ImSrzuUrstar)/.R->1;

Print["omega=",omega]; Print["wave#=",Y];
Print["rdi=",RDI]; Print["rdo=",RDO];
Print["adi=",ADI]; Print["ado=",ADO];
Print["normalizing factor =",C];

g1=Plot[Evaluate[{C^2*Pi*omega/(2*Im[Y])*ImSrrUrstar,
C^2*Pi*omega/(2*Im[Y])*ImSrzuUrstar,
C^2*Pi*omega/(2*Im[Y])*(ImSrrUrstar+ImSrzuUrstar)}],{R,0,1},
Frame->True,
FrameLabel->"Normalized Radius \(r/(r/a)\),
"Im \(\Im\{\{\text{Sigma}\}_1,k\}\)/\((\text{i}(\text{u}\_k\%*\))\)",
AspectRatio->1,
GridLines->Automatic,
EMBEDDED PILE CODE

Displacement and Power Routine (cont'd)

PlotRange->All,
PlotStyle->{{AbsoluteThickness[1],Hue[0.5],AbsoluteDashing[{4,2}]},
 AbsoluteThickness[1],Hue[0.3],AbsoluteDashing[{0.5,2}]},
 AbsoluteThickness[1],Hue[0]}
PlotLegend->"Im \(\\langle [\Sigma]_{rr}\rangle \rangle!\langle (u_{\_r\%*})\rangle\)",
"Im \(\\langle [\Sigma]_{rz}\rangle \rangle!\langle (u_{\_z\%*})\rangle\)",
"Im \(\\langle [\Sigma]_{rr}\rangle \rangle!\langle (u_{\_r\%*})\rangle+\langle (u_{\_z\%*})\rangle\)"
PlotLabel->"Power Distribution in the Pile",
LegendPosition->{-0.65,-1.5}, LegendSize->{1.5,0.5},
LegendLabel -> "Power", LegendLabelSpace ->0.5,
LegendSpacing->0,
LegendBorderSpace->0.5,
LegendOrientation ->Vertical, LegendShadow -> {0.02,.02};

g2=Plot[Evaluate[{Re[RadDispIn]*c,Im[RadDispIn]*c,Re[AxiDispIn]*c,
 Im[AxiDispIn]*c}],[R,0,1],
Frame->True,
FrameLabel->"Normalized Radius \(\langle r/\alpha\rangle\)","Displacements"},
PlotRange->All,
AspectRatio->1,
GridLines->Automatic,
PlotRange->All,
PlotStyle->{{AbsoluteThickness[1],Hue[0]},{AbsoluteThickness[1],Hue[0.8],
 AbsoluteDashing[{0.5,2}]},{AbsoluteThickness[1],Hue[0.7],
 AbsoluteDashing[{4,2}},{AbsoluteThickness[1],Hue[0.5],
 AbsoluteDashing[{{0.5,2,4,2}}]}
PlotLegend->"Radial-real","Radial-img","Axial-real","Axial-img"},
PlotLabel->"Normalized Displacements",
LegendPosition->{-0.5,-1.5},
LegendSize->{1.5,0.5},
LegendLabel -> "Displacements", LegendLabelSpace ->0.5,
LegendSpacing->0,
LegendBorderSpace->0.5,
LegendOrientation ->Vertical,
LegendShadow -> {0.02,.02};

Print["------------------------------------------ ",{i,1,50}]}
Figure A1  Normalized Power and Displacement Distributions for L(0,1)a Mode
Figure A2  Normalized Power and Displacement Distributions for L(0,1)b Mode
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NORTHWESTERN UNIVERSITY

An Experimental Model for Non-destructive Evaluation on Pile Foundations Using Guided Wave Approach

SUBMITTED TO THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS for the degree DOCTOR OF PHILOSOPHY Field of Civil Engineering

By

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ABSTRACT

An Experimental Model
for Pile Integrity Evaluation
Using Guided Wave Approach

Hsiao-chou Chao

The objective of this research is to extend the frequency range for surface reflection techniques for concrete piles using a three-dimensional guided wave approach. This study was conducted in four stages. First, a PC-controlled test system based on requirements found from results of numerical evaluation is developed and integrated. This test system is able to perform waveform generation, vibration measurement, multiple channel data acquisition, and various data processing and analyzing techniques such as digital filtering, Fast Fourier Transform (FFT), and Joint Time-Frequency Analysis (JTFA). Second, a series of small-scale prototype piles with diameters from 152mm (6 inch) to 457mm (18 inch) were constructed in the laboratory; these piles were later on embedded in the National Geotechnical Experimental Site at Northwestern University. These piles are either intact or contain a designed defect. Third, conventional impulse response tests were conducted on these piles. The results are evaluated by three-dimensional guided wave theory. Finally, the procedures to induce waves with a narrow band of frequency content into the piles were developed. The responses were evaluated by comparing the experimental and theoretical results. For interpreting the data using a guided wave approach, special techniques were developed for measuring the bulk shear wave velocity and identifying the mode attribute.
of acquired waveforms. Results of the research indicate the theoretical guided wave approach is applicable for interpreting the results for both the conventional impulse response tests and the newly developed frequency-controlled test. A new approach for detecting the locations and types of defect that cannot be identified by the impulse response approach was also proposed.

**Thesis advisor:**

Prof. Richard J. Finno
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Chapter 1

Introduction

In addition to use the results of laboratory or in-situs soil tests, the load capacity for a drilled shaft can be designed using the information provided from a load test on a full scale prototype drilled shaft. The load capacity of a in-service drilled shaft developed from the side resistance and end bearing capacity can be inspected by a proof load test. For practical application, however, the load test is only applied on representative drilled shafts for large projects, and may not be cost-effective for small to medium size project. The mechanism of side resistance is mobilized by the relative movement between the concrete surface and surrounding soils, while the bearing capacity is the resisting force provided from the bottom soil against the load transferred from the top of a drilled shaft through the concrete. Thus, it is not difficult to envision the load capacity of a drilled shaft is a function of the construction method. The actual bearing capacity of a drilled shaft could be different from the one predicted from the results of a load test on a representative drilled shaft because of the potential variation in the construction process. Furthermore, interpretation of the load test result usually does not involve the dimension of the pile and quality of the concrete. When a load test is conducted on a representative drilled shaft with an
unknown defect, for example, soil intrusion, the load capacity designed for pile-soil system could be underestimated because the additional deformation caused by the defect. To reduce the uncertainty of the expected load capacity and assure the quality of the concrete, the integrity of a newly installed drilled shaft needs to be evaluated by approaches other than the load test.

Non-destructive evaluation (NDE) techniques have been used for decades to provide quality assurance for drilled shafts and driven concrete piles. In particular, the surface reflection techniques including sonic-echo and impulse response methods have been used extensively to inspect the continuity and length of a newly installed drilled shaft. These methods can be conducted inexpensively and quickly for a great number of drilled shafts in a relatively short time span. The sonic echo method evaluates the integrity of a drilled shaft by interpreting the time domain wave trace of the measured surface vibrations triggered by the event of hammer impact. The impulse response method deals with the responses of both the impact force and surface vibrations. The integrity of a drilled shaft is evaluated by comparing the experimentally-derived frequency response of the drilled shaft and the numerically simulated frequency response based on construction data, design details, and soil parameters. The existence of anomalies in a drilled shaft can be detected by identifying discrepancy in the results between the experimental data and numerical results. The location and types of anomaly can thereafter be identified by finding the best matched input parameters using a numerical simulation program (Davis, 1994).

In addition to be a tool for assuring the integrity of a drilled shaft at different
stages of construction, the impulse response method can be used to evaluate unknown bridge foundations or existing deep foundations covered by pile cap by the multiple geophone approach developed by Gassman and Finno (1999). Gassman (1997) presented the results of the impulse response method on the drilled shafts in the National Geotechnical Experimental Site (NGES) at Northwestern University. These evaluations were conducted on drilled shafts with accessible and inaccessible pile heads using both single geophone and multiple geophone approach. Results of the evaluation show that the location and type of multiple defects in the test piles with accessible and inaccessible pile head can be detected. However, results of the conventional NDE method also reveal the following limitations in practical applications.

1. Unable to detect small size defects.

2. Unable to identify the defects close to each other or near the bottom of a drilled shaft.

3. Unable to distinguish certain types of defect; For example, necking and poor quality of concrete.

These limitations are the results of inherent properties of the stress waves generated by a hammer-impact. The wavelength of the propagation stress wave is large — at least larger than the diameter of the pile. The frequencies composed of the hammer-generated stress waves are low — less than several thousand hertz. To extend the applicability of the conventional non-destructive method, it is necessary to increase the useful frequency range such that the advantages of waves with smaller wavelength can be applied. A theoretical guided wave approach for evaluating the integrity of a drilled shaft was presented by Finno et al. 2001. This three-dimensional numerical
approach was developed by treating a drilled shaft as an embedded waveguide. Results of numerical evaluation show that special characteristics of a propagation wave such as the velocity, attenuation coefficient, power flux and displacement distribution may vary with frequency. The frequency ranges within which the waves have constant characteristics are considered preferable for conducting tests. The conventional NDE test is within one of these frequency ranges. Theoretical evaluation also shows the distribution of the vibration amplitude and power flux across the pile is not uniform for modes at higher frequencies, and thus the geometry of a defect can be determined by the reflections of various modes of waves. Therefore, the capability of the non-destructive methods for drilled shaft can be extended by taking advantage of the properties of stress waves with higher frequency contents.

The objective of this research is to evaluate the applicability of the theoretical guided wave approach on embedded cylindrical piles proposed by Hanifah (1999) and Finno et al. (2002) and to develop an experimental approach to identify the defects that are undetectable by the conventional impulse response method. The work presented in this dissertation was conducted in the following procedures. First, a PC-controlled test system able to perform multi-channel data acquisition (DAQ) and waveform generation was built based on preliminary numerical data analysis, and general soil and concrete properties. Second, a series of small scale prototype piles with diameters ranges from 152mm (6inch) to 457mm (18inch) and lengths from 0.91m (3ft) to 2.4m (7.9ft) were constructed in the laboratory and installed subsequently in the NGES site at Northwestern University. Third, the existence of guided waves in piles under traction-free and embedded boundary conditions was evaluated by both
the hammer-generated and the vibrator-generated frequency-controlled waves. The experimental results were then compared with the numerical results to verify the applicability of the numerical model. At last, the frequency-controlled method for identifying the pile tip and potential defects that is undetectable by the conventional impulse response method is proposed.

This dissertation encompasses seven chapters. Chapter 2 reviews the theoretical guided wave approach for cylindrical piles embedded in soils. The general frequency equation based on three-dimensional elastic theory developed by Finno et al. (2002) is presented. The first twelve branches axisymmetrical mode solutions for this general frequency equation are solved and plotted in frequency versus wave number. The group velocity, phase velocity, and attenuation coefficient are thereafter be derived from the dispersion relations. The power flux distribution, displacement distribution, and stress distribution across the radius are illustrated by selected modes. The frequency equation was developed based on the boundary condition that the propagation waves are excited by steady-state vibration source applied uniformly across the pile. The applicability of the steady-state solutions on the more realistic transient state boundary condition is analytically evaluated.

Chapter 3 discusses the practical considerations for designing the experiment based on the results of theoretical evaluation. The integrity inspection is performed by comparing the experimentally-observed results and numerically-derived results. The experiment involves (a)frequency range selection, (b)excitation of waves with a controlled frequency range, and (c)measurement of the group velocity, phase velocity,
and attenuation coefficient. The experimental results and theoretical results need to be converted to the same basis, dimensional or non-dimensional, such that the comparisons can be made directly. The conversion is conducted by applying the bulk shear wave velocity. Numerical evaluation shows that for each longitudinal branch there exists at least one solution at which the frequency is independent of all the pile and soil properties. This solution is called the universal mode. Theoretical evaluation reveals the bulk shear wave velocity can be determined from the universal frequency if which can be found experimentally.

Chapter 4 presents the development of the PC-controlled test system. This system is able to perform both the conventional impulse response test and the frequency-controlled guided wave test. It comprises three major parts: the portable computer, the vibration generation system, and the data acquisition system. The vibration generation system is composed of a vibration shaker, an impedance network, a power amplifier, and a virtual waveform generator. The data acquisition system is integrated by a number of accelerometers, signal conditioner, multi-function data acquisition boards, and a virtual oscilloscope. The portable PC controls the waveform generation and multiple channel data acquisition by the virtual waveform generator and the virtual oscilloscope programmed by the graphical application software LabView. The acquired data are processed by various technologies and stored in the portable PC for later analysis.

Chapter 5 summarizes the results of the interpretation for conventional impulse response tests on the small scale prototype piles using the 3-D guided wave theory.
The impulse response tests were conducted on piles with traction-free and embedded boundary conditions. The corresponding concrete and soil properties are determined based on in-situ tests done by Gassman (1997) and laboratory tests. The L(0,1) branch dispersion curves are developed from these concrete and soil parameters. The evidence of guide waves in conventional impulse response test was revealed from the results of the comparison between the experimental results and numerical results.

In chapter 6, the approach to control the frequency content of an excited waves is numerically evaluated. The performance of the vibration shaker is experimentally evaluated thereafter based on the results of numerical evaluation. The first three branches of dispersion curves are numerically developed from the pile and soil parameters determined in Chapter 5. An experimental approach for determine the bulk shear wave velocity based on the method to identify the universal frequency is presented. After the shear wave velocity is determined, the experimental results and numerical results can be compared directly. The theoretical concept for predicting the wave modes that can be excited by the proposed wave excitation method within a controlled frequency range is discussed. The forecast results are experimentally verified using the newly-developed mode identification technique. At last, the approach for evaluating the integrity of a drilled shaft and identifying the location and type of various defects using the frequency-controlled method, mode forecast and identification technique is presented.
Chapter 2

Theoretical Evaluation of Guided Wave in Embedded Piles

2.1 Introduction

A rigorous approach to analyze wave propagation in deep foundations is to treat the pile as a wave guide. Waves propagating in a pile can be considered guided waves which are combinations of longitudinal and shear waves that continually interact with the boundaries to produce a composite wave. Finno et al. (2002) presented the general frequency equation for a steady-state wave propagating along the infinitely long embedded piles. This solution is summarized herein to give an overview of the propagation modes that represent various motions that can be excited in a pile. Finno et al. (2002) solved the general frequency equation for the longitudinal modes, and presented the results in terms of dispersion, group velocity, attenuation, and modal shape. The overall evaluation of these longitudinal modes provides a baseline for the design of equipment and procedures to conduct non-destructive experiments that exploit the features of guided waves.
2.2 General Frequency Equation for Embedded Cylindrical Wave Guide

Following the general procedures for obtaining frequency equations of various wave guides (e.g. Zemanek 1971, Thurston 1978) general frequency equation for an infinitely long cylindrical pile embedded in soil was developed by Finno et al. (2002) based the basic dynamic equations of elasticity. A detail derivation is found in Hanifah (1999).

Cylindrical coordinates, \( r, z \) and \( \theta \), are chosen with the \( z \) direction coincident with the axis of the cylinder. Appropriate special functions are selected to account for the assumed steady-state wave propagation along the pile and the outward, decaying wave propagation in the soil. A solution to the wave equation of an isotropic elastic solid can be obtained in terms of a scalar and vector potential by the method of separation of variables. The solutions for the three components of displacement for piles, \( u_p \), and soil, \( u_s \), respectively, are of the form:

\[
\begin{align*}
    u_{rp} &= U_{rp}(r) \cos(n\theta)e^{i(\omega t - \xi z)} \\
    u_{\theta p} &= U_{\theta p}(r) \sin(n\theta)e^{i(\omega t - \xi z)} \\
    u_{zp} &= U_{zp}(r) \cos(n\theta)e^{i(\omega t - \xi z)} \\
    u_{rz} &= U_{rz}(r) \cos(n\theta)e^{i(\omega t - \xi z)} \\
    u_{\theta z} &= U_{\theta z}(r) \sin(n\theta)e^{i(\omega t - \xi z)} \\
    u_{zs} &= U_{zs}(r) \cos(n\theta)e^{i(\omega t - \xi z)}
\end{align*}
\]  

(2.1)

where \( t \) is the time, \( \omega \) is the angular frequency, \( \xi \) is the wave number, and \( n \) is equal...
to zero or an integer. The radial functions in Equation 2.1 are found to be:

\[
U_{tp}(r) = [A_{1p} \alpha_p Z_n'(\alpha_p r) - A_{4p} \xi Z_n'(\beta_p r)]
\]

\[
U_{\theta p}(r) = \left[ -A_{1p} \frac{n}{r} Z_n(\alpha_p r) + A_{4p} \frac{n}{r} \beta_p Z_n(\beta_p r) - A_{6p} \beta_p Z_n'(\beta_p r) \right]
\]

\[
U_{z p}(r) = i \left[ -A_{1p} \xi Z_n(\alpha_p r) - A_{4p} \beta_p Z_n(\beta_p r) \right]
\]

\[
U_{r s}(r) = \left[ A_{1s} \bar{\alpha}_s H_n^{(2)}(\alpha_s r) - A_{4s} \xi H_n^{(2)}(\beta_s r) + A_{6s} \frac{n}{r} H_n^{(2)}(\beta_s r) \right]
\]

\[
U_{\theta s}(r) = \left[ -A_{1s} \frac{n}{r} H_n^{(2)}(\alpha_s r) + A_{4s} \frac{n}{r} \beta_s H_n^{(2)}(\beta_s r) - A_{6s} \beta_s H_n^{(2)}(\beta_s r) \right]
\]

\[
U_{z s}(r) = i \left[ -A_{1s} \xi H_n^{(2)}(\alpha_s r) - A_{4s} \beta_s H_n^{(2)}(\beta_s r) \right]
\]

(2.2)

where \(Z_n(x)\) is the Bessel function of the first kind and order \(n\), and \(H_n^{(2)}\) is the Hankel function of the second kind of order \(n\). \(A_i\) are constants. The variable \(\alpha_j\) and \(\beta_j\) are defined as

\[
\alpha_j = \left( \frac{\omega}{c_{L_j}} \right)^2 - \xi_j^2
\]

(2.3)

\[
\beta_j = \left( \frac{\omega}{c_{T_j}} \right)^2 - \xi_j^2
\]

(2.4)

respectively. The subscript \(j\) denotes soil or pile. \(\alpha_j\) and \(\beta_j\) are the magnitude of \(\alpha_j\) and \(\beta_j\). The constants \(c_{L_j}\) and \(c_{T_j}\) are bulk wave velocities

\[
c_{L_j} = \sqrt{\frac{\lambda_j + 2\mu_j}{\rho_j}}
\]

(2.5)

\[
c_{T_j} = \sqrt{\frac{\mu_j}{\rho_j}}
\]

(2.6)

where \(\lambda_j\) and \(\mu_j\) are the Lame constants and \(\rho_j\) is the density. A Hankel function was selected to satisfy the condition that the outgoing waves in the soil are decaying
with distance in the radial direction, as the boundaries are located at infinity. By combining the strain-displacement and stress-strain relations with the displacement solution, one can derive the stress in the soil and pile to be:

\[
\sigma_{rr} = \left\{ \begin{array}{l}
A_{1p} \left\{ \begin{array}{l}
-\lambda_p \left( \bar{\alpha}_p^2 + \xi^2 \right) + 2\mu_p \left( \frac{n^2}{r^2} - \bar{\alpha}_p^2 \right) \\
Z_n(\bar{\alpha}_p r) - 2\mu_p \frac{\bar{\alpha}_p}{r} Z'_n(\bar{\alpha}_p r)
\end{array} \right. \\
+ A_{4p} 2\mu_p \xi \left( \frac{n^2}{\beta_p r^2} \right) Z_n(\beta_p r) + \frac{1}{r} Z'_n(\beta_p r) \\
+ A_{6p} 2\mu_p \frac{n}{r} \left[ \bar{\beta}_p Z_n(\beta_p r) - \frac{1}{r} Z_n(\beta_p r) \right]
\end{array} \right\} \cos(n\theta)e^{i(\omega t - \xi z)} \tag{2.7}
\]

\[
\tau_{\theta r} = \mu_p \left\{ \begin{array}{l}
A_{1p} \frac{2n}{r} \left[ \frac{1}{r} Z_n(\bar{\alpha}_p r) - \bar{\alpha}_p Z'_n(\bar{\alpha}_p r) \right] \\
A_{4p} \frac{2n\xi}{r} \left[ Z'_n(\beta_p r) - \frac{1}{\beta_p r} Z_n(\beta_p r) \right] \\
+ A_{6p} \left[ \frac{2\beta_p}{r} - Z_n(\beta_p r) - \left( \frac{2n^2}{r^2} - \beta_p^2 \right) Z_n(\beta_p r) \right]
\end{array} \right\} \sin(n\theta)e^{i(\omega t - \xi z)} \tag{2.8}
\]

\[
\tau_{\rho r} = i\mu_p \left\{ \begin{array}{l}
A_{1p} \left[ -2\xi \bar{\alpha}_p Z'_n(\bar{\alpha}_p r) \right]
+ A_{4p} \left[ (\xi^2 - \beta_p^2) Z_n(\beta_p r) \right] \\
+ A_{6p} \left[ -\frac{n\xi}{r} Z_n(\beta_p r) \right]
\end{array} \right\} \cos(n\theta)e^{i(\omega t - \xi z)} \tag{2.9}
\]

\[
\sigma_{\theta \theta} = \left\{ \begin{array}{l}
A_{1s} \left\{ \begin{array}{l}
-\lambda_s \left( \bar{\alpha}_s^2 + \xi^2 \right) + 2\mu_s \left( \frac{n^2}{r^2} - \bar{\alpha}_s^2 \right) \\
H''(\bar{\alpha}_s r)
\end{array} \right. \\
-2\mu_s \frac{\bar{\alpha}_s}{r} H''(\bar{\alpha}_s r) \\
+ A_{4s} 2\mu_s \xi \left( \frac{n^2}{\beta_s r^2} \right) H''(\beta_s r) + \frac{1}{r} H''(\beta_s r) \\
+ A_{6s} 2\mu_s \frac{n}{r} \left[ \bar{\beta}_s H''(\beta_s r) - \frac{1}{r} H''(\beta_s r) \right]
\end{array} \right\} \cos(n\theta)e^{i(\omega t - \xi z)} \tag{2.10}
\]
\[ \begin{align*}
\tau_{r\theta} &= \mu_s \left\{ A_1 \frac{2n}{r} \left[ \frac{1}{r} H_n(\bar{\alpha}_s r) - \alpha_s H_n'(\bar{\alpha}_s r) \right] \\
&\quad + A_4 \frac{2n\xi}{r} \left[ H_n^{(2)}(\bar{\beta}_s r) - \frac{1}{bar_s r} H_n^{(2)}(\bar{\beta}_s r) \right] \\
&\quad + A_6 \left[ \frac{2\bar{\beta}_s}{r} H_n^{(2)}(\bar{\beta}_s r) - \left( \frac{2n^2}{r^2} - \bar{\beta}_s^2 \right) H_n^{(2)}(\bar{\beta}_s r) \right] \right\} \cos(n\theta)e^{i(\omega t - \xi z)} \\
\tau_{rz} &= i\mu_s \left\{ A_1 \left[ -2\xi \bar{\alpha}_s H_n^{(2)}(\bar{\alpha}_s r) \right] \\
&\quad + A_4 \left[ (\xi^2 - \bar{\beta}_s^2) H_n'^{(2)}(\bar{\beta}_s r) \right] \\
&\quad + A_6 \left[ -\frac{n\xi}{r} H_n^{(2)}(\bar{\beta}_s r) \right] \right\} \cos(n\theta)e^{i(\omega t - \xi z)}
\end{align*} \] (2.11)

The lateral boundary conditions

\[ \begin{align*}
\begin{cases}
u_{rp} - u_{rs} \\
u_{\theta p} - u_{\theta s} \\
u_{zp} - u_{zs} \\
\sigma_{rrp} - \sigma_{rrs} \\
\tau_{r\theta p} - \tau_{r\theta s} \\
\tau_{rzp} - \tau_{rzs}
\end{cases} = 0
\end{align*} \] (2.13)

require the displacements and stresses at the pile-soil interface to be continuous. Application of the boundary conditions results in a homogeneous system of equations.
which is a product of a $6 \times 6$ coefficient matrix and a $6 \times 1$ constant matrix.

\[
\begin{pmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{pmatrix} \begin{pmatrix}
  A_{1p} \\
  A_{4p} \\
  A_{8p} \\
  A_{1s} \\
  A_{4s} \\
  A_{8s}
\end{pmatrix} = 0 \quad (2.14)
\]

where the $x_{mn}$ terms in the coefficient matrix are given as follows:

\[
\begin{align*}
  x_{11} &= \alpha_p Z'_n(\alpha_p a) \\
  x_{12} &= -\xi Z'_n(\beta_p a) \\
  x_{13} &= \frac{n}{\alpha} Z_n(\beta_p a) \\
  x_{14} &= -\alpha_s H^{(2)}_n(\alpha_s a) \\
  x_{15} &= \xi H^{(2)}_n(\beta_s a) \\
  x_{16} &= -\frac{n}{\alpha} H^{(2)}_n(\beta_s a)
\end{align*}
\]  

(2.15)
\[ x_{21} = -\frac{n}{a} \mathcal{Z}_n(\tilde{\alpha}_p a) \]
\[ x_{22} = \frac{n\zeta}{\tilde{\beta}_p a} \mathcal{Z}_n(\tilde{\beta}_p a) \]
\[ x_{23} = -\tilde{\beta}_p \mathcal{Z}'_n(\tilde{\beta}_p a) \]
\[ x_{24} = \frac{n}{a} \mathcal{H}^{(2)}_n(\tilde{\alpha}_s a) \]
\[ x_{25} = -\frac{n\zeta}{\tilde{\beta}_s a} \mathcal{H}'^{(2)}_n(\tilde{\beta}_s a) \]
\[ x_{26} = \tilde{\beta}_s \mathcal{H}^{(2)}_n(\tilde{\beta}_s a) \]

\[ x_{31} = -i\xi \mathcal{Z}_n(\tilde{\alpha}_p a) \]
\[ x_{32} = -i\tilde{\beta}_p \mathcal{Z}_n(\tilde{\beta}_p a) \]
\[ x_{33} = 0 \]
\[ x_{34} = i\xi \mathcal{H}^{(2)}_n(\tilde{\alpha}_s a) \]
\[ x_{35} = i\tilde{\beta}_s \mathcal{H}^{(2)}_n(\tilde{\beta}_s a) \]
\[ x_{36} = 0 \]
\begin{align}
x_{41} &= \left[-\lambda_p(\tilde{\alpha}_p^2 + \xi^2) + 2\mu_p \left(\frac{n^2}{a^2} - \tilde{\alpha}_p^2\right)\right] Z_n(\tilde{\alpha}_p a) - 2\mu_p \frac{\tilde{\alpha}_p}{a} Z'_n(\tilde{\alpha}_p a)

x_{42} &= 2\mu_p \xi \left[\left(\tilde{\beta}_p - \frac{n^2}{\beta_p a^2}\right) Z_n(\tilde{\beta}_p a) + \frac{1}{a} Z'_n(\tilde{\beta}_p a)\right]

x_{43} &= 2\mu_p \frac{n}{a} \left[\tilde{\beta}_p Z'_n(\tilde{\beta}_p a) - \frac{1}{a} Z_n(\tilde{\beta}_p a)\right]

x_{44} &= -\left[-\lambda_s(\tilde{\alpha}_s^2 + \xi^2) + 2\mu_s \left(\frac{n^2}{a^2} - \tilde{\alpha}_s^2\right)\right] H_n^{(2)}(\tilde{\alpha}_s a) + 2\mu_s \frac{\tilde{\alpha}_s}{a} H'_n^{(2)}(\tilde{\alpha}_s a)

x_{45} &= -2\mu_s \xi \left[\left(\tilde{\beta}_s - \frac{n^2}{\beta_s a^2}\right) H_n^{(2)}(\tilde{\beta}_s a) + \frac{1}{a} H'_n^{(2)}(\tilde{\beta}_s a)\right]

x_{46} &= -2\mu_s \frac{n}{a} \left[\tilde{\beta}_s H'_n^{(2)}(\tilde{\beta}_s a) - \frac{1}{a} H_n^{(2)}(\tilde{\beta}_s a)\right]

x_{51} &= \mu_p \frac{2n}{a} \left[\frac{1}{a} Z_n(\tilde{\alpha}_p a) - \tilde{\alpha}_p Z'_n(\tilde{\alpha}_p a)\right]

x_{52} &= \mu_p \frac{2n\xi}{a} \left[Z'_n(\tilde{\beta}_p a) - \frac{1}{\beta_p a} Z_n(\tilde{\beta}_p a) a\right]

x_{53} &= \mu_p \left[\frac{2\tilde{\beta}_p}{a} Z'_n(\tilde{\beta}_p a) - \left(\frac{2n^2}{a^2} - \tilde{\beta}_p^2\right) Z_n(\tilde{\beta}_p a)\right]

x_{54} &= -\mu_s \frac{2n}{a} \left[\frac{1}{a} H_n^{(2)}(\tilde{\alpha}_s a) - \tilde{\alpha}_s H'_n^{(2)}(\tilde{\alpha}_s a)\right]

x_{55} &= -\mu_s \frac{2n\xi}{a} \left[H'_n^{(2)}(\tilde{\beta}_s a) - \frac{1}{\beta_s a} \tilde{\alpha}_s H_n^{(2)}(\tilde{\beta}_s a)\right]

x_{56} &= \mu_s \left[\frac{2\tilde{\beta}_s}{a} H'_n^{(2)}(\tilde{\beta}_s a) + \left(\frac{2n^2}{a^2} - \tilde{\beta}_s^2\right) H_n^{(2)}(\tilde{\beta}_s a)\right]
\end{align}
\[ x_{61} = -i2\mu_p \xi \bar{\alpha}_p Z'_n(\bar{\alpha}_p a) \]
\[ x_{62} = i\mu_p \left( \xi^2 - \bar{\beta}_p^2 \right) Z'_n(\bar{\beta}_p a) \]
\[ x_{63} = -i\mu_p \frac{n \xi}{a} Z_n(\bar{\beta}_p a) \]
\[ x_{64} = i\mu_s 2\xi \bar{\alpha}_s H''_n(\bar{\alpha}_s a) \]
\[ x_{65} = -i\mu_s \left( \xi^2 - \bar{\beta}_s^2 \right) H''_n(\bar{\beta}_s a) \]
\[ x_{66} = i\mu_s \frac{n \xi}{a} H''_n(\bar{\beta}_s a) \]  
(2.20)

The only nontrivial solutions for the unknown matrix are those for which the determinant of the coefficient matrix is equal to zero. Therefore the determinant, \( D_1 \), is

\[
\begin{vmatrix}
  x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
  x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\
  x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \\
  x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} \\
  x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} \\
  x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & x_{66}
\end{vmatrix} = 0 \]  
(2.21)

The equation formed by Equation 2.21 is the frequency equation for harmonic wave propagation in an infinitely long embedded cylindrical pile. The solutions of this frequency equation can be plotted as dispersion curves showing the relation between frequency and wave number. Each dispersion curve is a branch of wave mode. The wave propagation in cylindrical piles can be considered as axisymmetric or antisymmetric. The axisymmetrical mode refers to the longitudinal mode in this dissertation, and is designated by the symbol \( L(0, m) \), where 0 represents \( n = 0 \) in Equation 2.21 and \( m \) represents the order of the branches. All the antisymmetrical modes are designated
by the symbol $F(n, m)$. The components in the unknown matrix in Equation 2.14 are the constants in the displacement equation 2.1 and stress distribution equation 2.7.

### 2.3 Longitudinal Mode Frequency Equation

If $n$ is set to zero in Equation 2.21, the general frequency equation can be simplified and decomposed into two sub-determinants, $D_2 \times D_3 = 0$, which are given by

$$
\begin{vmatrix}
  x_{11} & x_{12} & x_{14} & x_{15} \\
  x_{21} & x_{22} & x_{24} & x_{25} \\
  x_{31} & x_{32} & x_{34} & x_{35} \\
  x_{41} & x_{42} & x_{44} & x_{45}
\end{vmatrix} \times 
\begin{vmatrix}
  x_{23} & x_{26} \\
  x_{33} & x_{36} \\
  x_{43} & x_{46}
\end{vmatrix} = 0 \quad (2.22)
$$

The sub-determinant, $D_2 = 0$, represents the frequency equation for longitudinal modes, and only contains the non-zero displacement components $u_r$ and $u_\theta$. The sub-determinant, $D_3 = 0$, represents purely torsional modes. The only non-zero term is $u_\theta$. The longitudinal mode is rendered dimensionless using the substitutions from Equation 2.23 to Equation 2.27. The notations are introduced following Thurston
\begin{align*}
Y &= \xi a \tag{2.23} \\
V &= \bar{\alpha}_p a \tag{2.24} \\
W &= \bar{\alpha}_q a \tag{2.25} \\
X &= \bar{\beta}_p a \tag{2.26} \\
U &= \bar{\beta}_q a \tag{2.27}
\end{align*}

where $Y$ is the nondimensional wave number; and $V, W, X$, and $U$ are non-dimensional terms. Substituting Equations from 2.23 to Equation 2.27 into $D_2 = 0$ enables the dimensionless determinant to be expressed in a simplified form. The final form of the determinant representing the frequency equation for longitudinal modes is shown in Equation 2.28.
\[
\begin{align*}
&\begin{array}{c}
-V Z_1(V) \\
-Y Z_0(V) \\
\left\{ \begin{array}{c}
-\frac{\lambda_p}{\mu_p} (V^2 + Y^2) + 2V^2 \\
Z_0(V) + 2V Z_1(V)
\end{array} \right.
\end{array} \\
&\begin{array}{c}
Y Z_1(X) \\
-X Z_0(X) \\
2Y [X Z_0(X) - Z_1(X)] \\
2YV Z_1(V) \\
-\left[ Y^2 - X^2 \right] Z_1(x) \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
WH_1^{(2)}(W) \\
Y H_0^{(2)}(W) \\
\left\{ \begin{array}{c}
- \frac{\lambda_e}{\mu_p} (W^2 + Y^2) \\
+ 2\frac{\mu_e}{\mu_p} W^2 \\
H_0^{(2)}(W) \\
+ 2\frac{\mu_e}{\mu_p} W H_1^{(2)}(W)
\end{array} \right.
\end{array} \\
&\begin{array}{c}
-2\frac{\mu_e}{\mu_p} Y \\
\left[ U H_0^{(2)}(U) - H_1^{(2)}(U) \right]
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\begin{array}{c}
\frac{H_e}{\mu_p} [Y^2 - U^2] H_1^{(2)}(U)
\end{array}
\end{align*}
\]

\[= 0 \ (2.28)\]
2.4 Dispersion Curves of Longitudinal Mode Propagation Waves

Dispersion wave propagation means the wave number is a function of frequency. Equation 2.28 was solved numerically to define the dispersion curves for the longitudinal modes.

The nondimensional frequency, $\Omega$, is related to the angular frequency $\omega$:

$$\Omega = \frac{\omega a}{c_{Tp}}$$  \hspace{1cm} (2.29)

The other nondimensional terms in Equation 2.28 can be expressed as:

$$V^2 = \left(\frac{c_{Tp}}{c_{Lp}}\right)^2 \Omega^2 - Y^2$$  \hspace{1cm} (2.30)

$$X^2 = \Omega^2 - Y^2$$  \hspace{1cm} (2.31)

$$W^2 = \left(\frac{c_{Tp}}{c_{Ls}}\right)^2 \left(\frac{c_{Ts}}{c_{Ls}}\right)^2 \Omega^2 - Y^2$$  \hspace{1cm} (2.32)

$$U^2 = \left(\frac{c_{Tp}}{c_{Ts}}\right)^2 \Omega^2 - Y^2$$  \hspace{1cm} (2.33)

Based on fundamental elasticity, the following expressions can be derived:

$$\frac{c_{Tz}}{c_{Lz}} = \left[\frac{1 - 2\nu_j}{2(1 - \nu_j)}\right]^{\frac{1}{2}}$$  \hspace{1cm} (2.34)

$$\left(\frac{c_{Tp}}{c_{Ts}}\right)^2 = \frac{\mu_p}{\mu_s} \times \frac{\rho_s}{\rho_p}$$  \hspace{1cm} (2.35)

$$\lambda_j = \frac{2\nu_j}{1 - 2\nu_j}$$  \hspace{1cm} (2.36)

$$\frac{\lambda_s}{\mu_p} = \frac{2\nu_s}{1 - 2\nu_s} \times \frac{\mu_s}{\mu_p}$$  \hspace{1cm} (2.37)
All the terms in nondimensional frequency equation, except the frequency and the wave number, can be determined if the pile properties, $\mu_p$, $\rho_p$ and $\nu_p$, and the soil properties, $\mu_s$, $\rho_s$ and $\nu_s$, are defined. Given these properties, the nondimensional frequency equation has only two unknowns, the nondimensional frequency, $\Omega$, and nondimensional wave number, $\xi a$. Figure 2.1 shows a graphical representation of the numerical solution in terms of non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$ for typical values of pile and soil properties. This solution was found using numerical procedures described by Hanifah (1999). The assumed values of concrete and soil properties are listed in Table 2.1.

The relative magnitudes of the shear and longitudinal wave velocities in the pile

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Concrete</th>
<th>Soft/Loose Soil</th>
<th>Hard/dense Soil</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poisson's ratio</td>
<td>0.18</td>
<td>0.3</td>
<td>0.3</td>
</tr>
<tr>
<td>Density (kg/m$^2$)</td>
<td>2400</td>
<td>1800</td>
<td>2000</td>
</tr>
<tr>
<td>Young's Modulus (MPa)</td>
<td>30000</td>
<td>105</td>
<td>755</td>
</tr>
<tr>
<td>Shear Modulus (MPa)</td>
<td>13000</td>
<td>40</td>
<td>290</td>
</tr>
<tr>
<td>Longitudinal Wave Velocity (m/s)</td>
<td>3725</td>
<td>280</td>
<td>710</td>
</tr>
<tr>
<td>Density Ratio, $\frac{\rho_p}{\rho_s}$</td>
<td>1.3</td>
<td>1.2</td>
<td></td>
</tr>
<tr>
<td>Shear Modulus Ratio, $\frac{\mu_p}{\mu_s}$</td>
<td>325</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

and soil are generally in the order as $c_{Lp} > c_{Tp} > c_{Ls} > c_{Ts}$. For these relative magnitudes, the wave number must be complex to satisfy the selected Hankel function so that the radially outward propagation waves decay at infinite distance. Numerical analysis (Finno et al., 2002) shows that the real part of the wave number is inde-
Figure 2.1: Dispersion Curves for the first twelve longitudinal branches based on the "loose" soil condition where $\nu_p = 0.23$, $\frac{\rho_p}{\rho_s} = 1.3$ and $\frac{\mu_p}{\mu_s} = 325$. $\text{Re}(\xi a)$: the real part of wave number, $\text{Im}(\xi a)$: the imaginary part of wave number.
pended of the stiffness of the surrounding soil. It implies that the wave propagation velocities are functions of the properties only of the pile. The imaginary part of the wave number represents the geometrical attenuation caused as the wave energy radially leaks into the surrounding soil. The wave energy was carried by the group waves which propagate along the cylindrical pile. The complex wave numbers in turn represents how the waves both propagate and attenuate with distance.

2.5 Group Velocity

The phase and group velocities can be derived from the dispersion curves. Consequently, waves of different lengths propagate with different phase and group velocities. The group wave is an important wave from an experimental view, because the energy moves with it.

The relationship between wave number, phase velocity and group velocity are illustrated by a simple example in this section. A solution consists of just two waves, instead of a continuum of waves of all wave numbers, wherein nearby wave numbers are considered, \( \xi_1 = \xi + \Delta \xi \) and \( \xi_2 = \xi - \Delta \xi \), where \( \Delta \xi \) is very small. The corresponding frequencies are \( \omega_1 = \omega + \Delta \omega \) and \( \omega_2 = \omega - \Delta \omega \), respectively. Assume the solution is

\[
    u(z, t) = A \cos(\omega_1 t - \xi_1 z) + A \cos(\omega_2 t - \xi_2 z) \tag{2.38}
\]
Using the trigonometric addition formula for cosines,

\[ \cos \theta + \cos \phi = 2 \cos \left( \frac{\theta + \phi}{2} \right) \cos \left( \frac{\theta - \phi}{2} \right) \]

we obtain

\[
\begin{align*}
  u(z, t) &= 2A \cos(\omega t - \xi z) \cos(\Delta \omega t - \Delta \xi z) \\
  &= 2A \cos[\xi(\frac{\omega}{\xi} t - z)] \cos[\Delta \xi(\frac{\Delta \omega}{\Delta \xi} t - z)] \\
  &= 2A \cos[\xi(c_p t - z)] \cos[\Delta \xi(c_g t - z)]
\end{align*}
\]

(2.39)

where \( c_p \) is phase velocity, and \( c_g \) is group velocity, respectively.

Figure 2.2 illustrates the solution to Equation 2.38 in terms of the group and phase velocity at \( z = 0 \) and \( \delta \omega = \frac{1}{20} \omega \). The solution is a product of two waves whose spatial behavior differs. One wave number is \( \xi \), nearly the same as the original two wave numbers, while the other has a very small wave number, \( \Delta \xi \), corresponding to a very large wave length. The two waves alternatively interfere constructively or destructively. The long wave acts as an envelope of the short waves. Between each zero of the long envelope, there appears a group of short waves. The short wave propagates with the phase velocity, \( c_p \), which is equal to \( \frac{\omega}{\xi} \) while the long wave propagates with the group velocity, \( c_g \), which is equal to \( \frac{\Delta \omega}{\Delta \xi} \). The group velocity, \( c_g \), is an important physical quantity in the viewpoint of measurement because packets of wave energy move at this velocity rather than the phase velocity.

The dispersion curves for longitudinal mode of an embedded pile exhibit a similar
Figure 2.2: Group and Phase Velocity. Illustrated by an example in which
\[ u(z, t) = \cos(20\omega t)\cos(\omega t), \text{ at } z = 0 \]
dispersion relation between the wave number and frequency. The real part of the wave number, \( \xi \), essentially represents the wave number along the axis, in our case the \( z \) direction, of a cylindrical pile. The corresponding wave length is \( \frac{2\pi}{k} \), and hence the wave number is the number of periods in \( 2\pi \) distance. In a guided wave experiment, wave number is not a physical quantity that can be measured directly. However, the group velocity can be computed conventionally by dividing the propagation distance by the measured time interval.

As demonstrated in Figure 2.3, the group velocity for the \( L(0,1) \) branch is approximately equal to phase velocity in the low frequency range but is lower than phase velocity in intermediate frequency range. At high frequency, both the phase and group velocities approaches the Rayleigh wave velocity.

The impulse response and sonic echo tests are examples of conventional NDE tests wherein it is assumed that phase velocity is equal to group velocity. As illustrated in Figure 2.4, the phase and group wave move together so that no carried waveform is observed within the major waveform envelopes. The frequency range of impulse response test for deep foundations are generally in the non-dispersive low frequency range where the phase and group waves move together.

For \( L(0,2) \) and higher modes shown in Figure 2.5, the group waves propagate more slowly than the corresponding phase wave in the intermediate frequency range. At high frequency, both phase and group velocities approach the shear wave velocity of the wave guide.
Figure 2.3: First Branch of Longitudinal Modes: Group and Phase Velocity.
Figure 2.4: A typical result of impulse response test shows no waveform is "carried" by the group velocity "envelopes". Figure Source: Shaft no.2 NGES, Amherst, MA
Figure 2.5: 2nd and Higher Branches of Longitudinal Modes: Group and Phase Velocity.

Generally, a constant velocity wave that propagates over a wide frequency range is preferred in a guided wave experiment because less frequency control is needed for the input source. Based on this criteria, the potential testing frequency range for a guided wave experiment should be within the plateau near the peaks of the second and higher longitudinal mode group velocity dispersion curves, the low frequency range for the first longitudinal mode and the asymptotic high frequency range for
every longitudinal mode.

As discussed in Section 2.4, the real part of the dispersion curve is essentially independent of the surrounding soil condition. The group propagation velocity is also independent of the surrounding soil condition because it is derived directly from the dispersion curve. The imaginary part of the wave number shows the energy leaking radially into the surrounding soil is frequency dependent, and implies that the guided waves are capable of propagating at certain frequency ranges while not at others.

2.6 Longitudinal Mode

Attenuation Dispersion Curves

The group velocity, i.e. the propagation velocity of a guided wave, is independent of the surrounding soil condition and essentially is a function of the elastic properties of the pile. Although the propagation velocity does not vary with surrounding soil stiffness, the energy it is able to carry is frequency dependent.

Figure 2.6 shows the nondimensional attenuation ($\xi_a$ vs. $\Omega$) and nondimensional group velocity dispersion curves for the soft/loose soil in Table 2.1. For L(0,1) mode, the two dispersion curves exhibit similar trends in the low frequency range ($\Omega < 2$). The constant propagation velocity plateau corresponds to the constant low attenuation plateau and implies insignificant dispersion effects on the L(0,1) branch in this low frequency range. The attenuation increases drastically after $\Omega > 2$ while the corresponding propagation velocity decreases sharply. Although wave propagation
velocity for $L(0,1)$ branch asymptotically approaches the Rayleigh wave velocity at high frequencies, the corresponding high attenuation prevents the mode from propagating any significant distance.

For $L(0,2)$ branch shown in Figure 2.6 as a dashed line, the group velocity and attenuation exhibit a plateau near $\Omega = 5$, implying that the $L(0,2)$ branch at this frequency is able to propagate at an approximately constant and maximum value. For $L(0,3)$ and higher branches, the velocity peaks do not line up with the corresponding attenuation peaks at the same frequency. At higher frequencies, the attenuation of $L(0,2)$ and higher modes asymptotically approach a minimum value while their corresponding propagating wave velocities also asymptotically approach the shear wave velocity. This phenomena leads to an inference that the guided wave test can be conducted at this higher frequency range because of the constant propagation velocity and low attenuation. However, the inhomogeneous nature of a waveguide made of concrete, i.e. a drilled shaft or driven concrete pile, causes a significant amount of material damping that prevents the higher frequency waves from propagating large distances. Material damping is caused by interparticle wave reflecting and scattering and the divergence of wave front (Gaydeck, 1992). Another issue needs to be addressed in developing an experiment for guided waves is the relative difficulty to excite a wave mode at the frequency range of interested. The concept of modal shapes is introduced as a means to evaluate this factor.
Figure 2.6: The attenuation and velocity dispersion curves for "loose/soft" soils. The phase velocity, group velocity, and attenuation coefficient of the selected modes $L(0,1)a$, $L(0,2)a$, $L(0,3)a$, and $L(0,4)a$ are marked in each of the dispersion curves.
2.7 Modal Shape

The modal shape describes the radial and axial motions induced by longitudinal wave propagation. Modal shapes for longitudinal wave propagation have been published by Finno et al. (2001). General modal shape for a steady-state wave at \( z = 0 \) inside a cylindrical wave guide can be derived by substituting the given frequency and wave number into the coefficient matrix (Equation 2.14) and solving the corresponding null vector which contains the values of \( A_{1p} \) and \( A_{4p} \) (Equation 2.14):

\[
\begin{align*}
    u_{rp} &= \frac{1}{a_i}[-A_{1p}VZ_1(VR) + A_{4p}YZ_1(XR)] \quad (2.40) \\
    u_{zp} &= -\frac{i}{a_i}[A_{1p}YZ_0(VR) + A_{4p}XZ_0(XR)] \quad (2.41) \\
    \bar{\sigma}_{rrp} &= \frac{\mu_p}{a^2} \left( A_{1p} \left\{ - \left[ \frac{\lambda_p}{\mu_p} (V^2 + Y^2) + 2V^2 \right] Z_0(VR) + \frac{2V}{R} Z_1(VR) \right\} \right) \quad (2.42) \\
    \bar{\tau}_{rsp} &= \frac{i\nu_p}{a^2} \left\{ A_{1p} \left[ 2YZ_1(VR) \right] - A_{4p} \left[ (Y^2 - X^2) Z_1(XR) \right] \right\} \quad (2.43)
\end{align*}
\]

where \( u_{rp} \) represents the radial modal shape inside the pile, \( u_{zp} \) represents the axial modal shape inside the pile, \( \bar{\sigma}_{rrp} \) and \( \bar{\tau}_{rsp} \) represent the axial and shear stress distribution, respectively.

Hanifah (1999) evaluated seventeen selected points located in the local maximum and minimum of the group velocity dispersion curves from the L(0,1) to L(0,5) branches. The modal shapes of interested were normalized by a constant, \( g \), that was derived by equating the power entering the top of a section of a pile to that leaving.
the pile through both the bottom and sides of the pile section:

\[ g = \sqrt{\frac{2\xi a}{\pi \Omega c_T a Im[\sigma_{rr}(r)u_r^*(r) + \tau_{rz}(r)u_z^*(r)] n_r|_{r=a}}} \]

(2.44)

where the axial power in the mode is expressed in terms of two radial components, a shear component given by \( \tau_{rz}(r)u_z^*(r) \) and a normal component given by \( \sigma_{rr}(r)u_r^*(r) \). The * terms represent the complex conjugates of the displacement components. These radial components of power relate to the modal energy leaking radially into the soil. The normalized modal shapes in the pile with respect to unit power flowing through a cross-section of the pile are then obtained by multiplying Equation 2.40 and 2.41 by \( g \). The normalized shapes from different modes can then be compared directly to evaluate the relative amplitude of the axial and radial components.

The modal shape evaluation shows that a modal point on a higher branch has a more complex shape than one on a lower branch. This trend is illustrated in Figures 2.7 and 2.8 where the selected modes are those having peak group velocities in their respective branch. The displacement profile of these modes oscillate and the oscillation increases with increasing order of the branch. The more complex modal shapes at higher branches implies even though the propagation velocity is constant with relatively low geometric attenuation, they will not be easy to excite because of their complex shapes. The high material damping also adversely restricts wave propagation. The modal shape for the L(0,1) branch in the low frequency range is close to a plane wave, and thus coincides with the assumption of 1-D wave propagation in an impulse response test. The plateau of L(0,2) branch has the most simple shape.
compared to the other branches' plateaus. Considering the combined effects of group velocity propagation, attenuation, material damping and modal shape, the $L(0,1)$ branch in the low frequency range and the $L(0,2)$ branch near its plateau frequency are considered as good candidate modes to be excited in a guided wave experimental design.

2.8 Application of Steady-state Wave Solution on the Transient-State Waves

The theoretical evaluation of the guided wave approach is based on the premise that a steady-state wave propagates in an infinitely long embedded cylindrical wave guide. In practical non-destructive applications, drilled shafts are neither infinitely long nor are subjected to a single frequency propagation wave. Current experimental methods apply input waves of either broad or narrow frequency band to a finite length wave guide. The extra boundary conditions induced by the transient input waveform and finite length wave guide make the derivation of the real solution impractical. Thurston (1978) indicated that many unnecessary details appear in the derivation of transient-state solution. Because a transient-state propagation wave can be considered as superposition of steady-state waves, it is assumed certain degree of agreement exists between the solution of steady-state and transient-state propagation waves. A conceptual evaluation on this assumption is conducted herein.

The Fourier integral theorem expresses a transient wave in terms of superimposed steady-state components with a spectrum of frequencies. If there is dispersion, each
Figure 2.7: Modal Shapes: (a) L(0,1)a, (b) L(0,2)a
Figure 2.8: Modal Shapes: (c) $L(0,3)a$, (d) $L(0,4)a$
Table 2.2: A One-dimensional Example of Deriving Transient-State Solution from Steady-State Solution

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady-State</td>
<td>$h(x)e^{i\omega t}$</td>
</tr>
<tr>
<td>Transient-State</td>
<td>$h(x)f(t)$</td>
</tr>
</tbody>
</table>

Component propagates with the phase velocity corresponding to its frequency. As a result, a transient wave tends to disperse as it propagates. A simple example shown in Table 2.2 illustrates the derivation of transient-state solution from one-dimensional steady-state solution. The steady-state boundary condition is assumed to be $h(x)e^{i(\omega t)}$, where $h(x)$ is a position function. The resulting steady-state solution is represented by $g(x, \omega)e^{i\omega t}$. Let $\tilde{f}(\omega)$ be the Fourier transform of $f(t)$. For a given value of $\omega$, multiply the steady-state boundary condition by $\frac{1}{2\pi} \tilde{f}(\omega)$ to obtain

$$\frac{1}{2\pi} \tilde{f}(\omega)h(x)e^{i\omega t} \quad (2.45)$$

The resulting steady-state solution is

$$\frac{1}{2\pi} \tilde{f}(\omega)g(x, \omega)e^{i\omega t} \quad (2.46)$$

By integrating the boundary condition with respect to frequency from $-\infty$ to $\infty$, the transient-state boundary condition is obtained:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)h(x)e^{i\omega t} d\omega = h(x)f(t) \quad (2.47)$$
The solution is

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) g(x, \omega) e^{i\omega t} d\omega \]  \hspace{1cm} (2.48)

This is Fourier superposition. By integrating, we have superimposed steady-state solutions to obtain the solution of the transient problem.

The agreement between the steady-state and transient-state solutions can be further illustrated by an impulse response transient boundary condition. Assume \( f(t) = \delta(t) \), where \( \delta(t) \) is a delta function, then \( \tilde{f}(\omega) = 1 \) and the resulting impulse response solution is

\[ \tilde{f}(\omega) g(x, \omega) = g(x, \omega) \]  \hspace{1cm} (2.49)

and the solution shows that the solution of the impulse boundary condition problem has the same complex amplitude as the steady-state solution.

The feasibility of applying steady-state solution on transient-state boundary condition problems has been illustrated by a simple example as above. Interpretation of impulse response test on drilled shafts using a guided wave approach will be studied in Chapter 4. Transient waves applied for mode excitation will be discussed in next chapter.

2.9 Summary

The general frequency equation for a steady-state wave propagating in an embedded cylindrical waveguide developed by Finno et al. (2002) is summarized in Section 2.2.
The frequency equation for longitudinal wave propagation is developed by setting the order of special functions in general frequency equation to zero so that the antisymmetric terms are eliminated. The normalized longitudinal mode frequency equation is listed in Equation 2.28.

The solutions of the longitudinal mode frequency equation are summarized, evaluated and presented from Section 2.4 to Section 2.7 in terms of dispersion, group velocity, attenuation and modal shape.

Transient-state waves are often used in practical non-destructive evaluation instead of steady-state waves. Transient-state waves can be considered as the superposition of steady-state waves. A theoretical example of using the solutions of steady-state waves on transient-state boundary condition is summarized in Section 2.8.

Considering the combined effects of group velocity propagation, attenuation, material damping and modal shape, the L(0,1) branch in the low frequency range and the L(0,2) branch near its plateau frequency are considered as good candidate modes to be excited in a guided wave experimental design.
Chapter 3

Experimental Design Consideration

3.1 Statement of the Problem

3.1.1 Guided Waves

In wave propagation problems, local disturbances decay rapidly in three-dimensional space. For practical application, energy needs be confined to one or two dimensions so that the information contained within the waves in three-dimensional world can be effectively retrieved. A wave guide is an extended body with a cross section of finite dimensions within which stress waves propagate in the extended direction despite of the incident and reflect at the boundaries (Achenbach, 1973). Typical wave guides, such as long hollow tubes, circular bars, or plates, permit wave propagation in one or two dimensions. Based on this concept, an embedded pile can be treated as a wave guide for a transient pulse introduced through the pile head. The continuing interaction of the excited waves with the boundaries produces the frequency-dependent guided waves.
3.1.2 Longitudinal and Flexural Waves

Longitudinal waves are axially symmetric waves characterized by displacement components in the radial and axial directions. Conventional surface reflection techniques for deep foundation such as sonic-echo or impulse response tests are essentially guided wave tests conducted at the low frequencies where the group velocity is non-dispersive and the stress wave is of the first longitudinal branch.

Flexural waves for traction free columns can be generated by transverse impacts on the circumferential boundaries (Lin and Sansalone, 1992), eccentric axial transient vibration from the end surface (Zemanek, 1971), or antisymmetric surface loading (Shin and Rose, 1998). However, flexural modes are not easy to excite in an embedded pile system in which vibration excitation and measurement are both conducted in the pile head. Therefore, the excitation and application of flexural wave is not include in this research.

3.1.3 Vibration Excitation and Measurement

In conventional surface reflection techniques applied to deep foundations, interpretation of test results are made assuming plane waves propagate along the pile (Davis and Dunn, 1974). All the points on a testing surface are moving in phase, and the excited displacement profile is uniform. Therefore, regardless the measurement position, the frequency response of a pile under test can be calculated by dividing the responding surface vibration with the input force even when they are not measured at the same point. The frequency responses of a pile caused by a steady state excitation
force $f(t)$

$$f(t) = F_0 e^{i\omega t}$$  \hspace{1cm} (3.1)

can be expressed in forms listed in Table 3.1 (after McConnell 1995), where $F_0$ is the excitation vector. The transformation between different frequency responses can be performed by time integral or derivative depending on the physical quantity measured. The frequency response analysis on the impulse response results is capable of identifying multiple changes of cross section or an equivalent defect. For example, the expected mobility of a drilled shaft can be simulated by the construction data. The integrity of the tested piles can thus evaluated by comparing the real and simulated mobilities. If anomalous response is identified, the locations and types of anomaly can be evaluated by a process of rematching the simulation to the expected result. This analysis is made based on the assumption that for a hammer-impact generated vibrations, the average mobility is constant in a broad frequency band being excited. However, for higher frequency application, the assumptions for impulse response analysis are not valid. The surface vibrations being excited are not uniformly distributed
across the pile. The frequency range and input energy level required for conducting guided wave experiments are usually beyond the capability of the modal hammer. The mobility for a specific branch of propagation wave may not a constant over the frequency range of interest. Multiple modes of propagation waves can be excited. Therefore, both the input vibration excitation and response measurements have to be conducted with an approach different from the impulse response test.

3.1.4 Objectives

The objective of this research is to evaluate the feasibility of extending the frequency range for which the integrity of pile foundations can be non-destructively evaluated. The experiments will be designed by considering mode characterization and mode selection. Mode characterization focuses on the verification of theoretical modal shape, propagation velocity and geometrical attenuation. Potentially acceptable modes for defect detection will be numerically identified and selected for developing the new non-destructive guided wave techniques. The feasibility of using the designed guide wave approach is evaluated through testing on prototype foundations. Factors affecting the design of the guided wave experiments include cutoff frequency, the measurement of phase and group velocity, geometrical attenuation coefficient, modal shapes and universal modes.
3.2 Design Considerations

3.2.1 Cutoff Frequency

Often it is desirable to design a guided wave test such that only one wave propagates at a given frequency. Otherwise the signal is more complicated, composed of two or more waves with different wave lengths but the same frequency. As shown in Figure 3.1 of the numerically-developed dispersion curves based on the parameters: \( \frac{\rho_p}{\mu_s} = 1.3, \frac{\mu_p}{\mu_s} = 325 \) and \( \nu_p = 0.23 \), only the lowest branch, \( L(0, 1) \) within this frequency range \( 0 < \Omega_f < \Omega_{c2}' \) can satisfy this criteria. Conventional surface reflection techniques such as sonic-echo and impulse response test are both conducted within this frequency range. The frequency of \( \Omega_{c2}' \) is considered as the frequency at which the \( L(0, 2) \) mode starts to propagate, beyond which two or more than two wave modes could be excited at the same frequency. In order to eliminate the complexity caused by multiple mode propagation, selection of frequency range based on attenuation coefficient will be conducted at frequencies lower than \( \Omega_{c2}' \).

Except for the cutoff frequencies, numerical analysis shows that the real part solutions of the frequency equation are independent of the elastic properties of the surrounding soils. Cutoff frequency is defined as the frequency below which the wave is evanescent. For traction-free piles, the cutoff frequency corresponds to the frequency at zero wave number (infinity wave length or phase velocity). However, as illustrated in Figure 3.1, the cutoff frequencies for embedded pile system do not necessarily correspond to a zero wave number. The \( L(0, 2) \) and \( L(0, 4) \) dispersion curves for the embedded piles show that each wave starts to propagate at a non-trivial \( \xi a \) instead.
Figure 3.1: Cutoff frequencies for both free an embedded piles: Poisson's ratio, $\nu = 0.23$. 
Table 3.2 lists the cutoff frequencies for same concrete pile with traction-free and embedded boundary conditions, respectively. These cutoff frequencies can be used as an index to distinguish whether or not a wave mode of interest could be excited at a given frequency range and to determine the potential frequency range for a given branch of longitudinal waves.

Table 3.2: Non-dimensional Cutoff Frequencies for Concrete Pile with Poisson's ratio $\nu = 0.23$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Traction Free Piles $\Omega_c$</th>
<th>Embedded Piles $\Omega_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(0, 1)$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$L(0, 2)$</td>
<td>3.44</td>
<td>2.88</td>
</tr>
<tr>
<td>$L(0, 3)$</td>
<td>3.84</td>
<td>3.24</td>
</tr>
<tr>
<td>$L(0, 4)$</td>
<td>7.02</td>
<td>7.42</td>
</tr>
<tr>
<td>$L(0, 5)$</td>
<td>9.10</td>
<td>8.60</td>
</tr>
</tbody>
</table>

### 3.2.2 Measurement of Group and Phase Velocity

In guided wave applications, cutoff frequency provides one way to evaluate if there are two or more waves excited at a given frequency. The group wave carrying the energy of the propagating wave can be measured directly from the waveforms recorded in the time domain. When only one mode of a propagating wave is excited at a frequency at which two or more modes potentially can be excited, the mode attribute of the excited mode can be identified by comparing the expected group wave propagation
velocity with the measured value. The phase velocity, however, cannot be measured directly from the time-domain waveforms. Nevertheless, the phase velocity can be estimated by the harmonics of the fundamental frequency based on the principle that the resonance of a pile occurs when the integer number of a half wavelength is equal to the pile length.

**Group Velocity Measurement**

In impulse response tests, the input force and induced vibration are both measured on the pile head. The wave propagation time $\Delta t$ between reflections can be found directly from measured waveforms in the time domain. The propagating waveforms represent superimposed group waves so that the bulk wave velocity can be calculated in the manner

$$c = \frac{2L}{\Delta t}$$

(3.2)

where $2L$ is the predetermined propagation distance which is generally twice the pile length. The calculated wave velocity represents the average group velocity of all the wave modes in the excited frequency band. For the L(0,1) mode propagation waves in the frequency range excited by a hammer-impact, the group velocity is approximately constant such that the group velocity of individual mode is equal to the average group velocity.

For practical application, it is preferable to perform group velocity measurement when a wave composed of modes having approximately constant or non-dispersive group velocities can be excited.
Phase Velocity Measurement

Assume the propagation modes in a finite length pile are the same as those in an infinite pile, and suppose an incident longitudinal wave is reflected without phase change as a wave of the same type but travelling in the opposite direction. The phase velocity of longitudinal waves in finite length pile can be measured in the frequency domain by means of a resonant method. The evaluation of a resonant solution for an embedded pile is summarized as below.

Rewrite the expression for the radial distribution of axial displacement in the pile (equation 2.41) as

\[ u_{zp} = \frac{i}{a} \left[ A_{1p}Z_0(ZR) + A_{4p}XZ_0(XR) \right] e^{i(\omega t - \xi z)} \]  \hspace{1cm} (3.3)

where the exponential term, $e^{i(\omega t - \xi z)}$, represents a harmonic wave propagating with a phase velocity, $\frac{\omega}{\xi}$, downward in the axial $z$-direction. Because the exponential term is independent of the radial components, at a distance along the radius of a pile with given material properties, the steady-state solution of $u_{zp}$ can be simplified as

\[ u_{zp} = U(\omega, \xi) e^{i(\omega t - \xi z)} \]  \hspace{1cm} (3.4)

where $U(\omega, \xi)$ is the amplitude function and will be constant for a given mode. As an example suppose that the pile in Figure 3.2 is subject to the steady-state displacement boundary condition

\[ u = U e^{i\omega t} \]  \hspace{1cm} (3.5)
at the pile head where $z = 0$, and the pile is bonded to a stiff bottom which is considered fixed, such that

$$u_{zp}(L, t) = 0$$  \hspace{1cm} (3.6)

The resulting steady-state motion can be approximately estimated in the following manner. The steady-state boundary condition induces a steady-state wave propagating in the positive $z$ direction. Because of the reflection at the bottom boundary, a steady-state wave will also propagate in the negative $z$ direction. The superimposed solution can be assumed in the form:

$$u_{zp} = U_T e^{i(\omega t - \xi z)} + U_R e^{i(\omega t + \xi z)}$$  \hspace{1cm} (3.7)

where $U_T$ is the amplitude of the propagating transmitted wave and $U_R$ is the amplitude of the propagating reflected wave. Substitute Equation 3.7 into the boundary
conditions at the pile head and toe, yields

\[ U_T + U_R = U \]  \hspace{1cm} (3.8)

\[ U_T e^{-\xi L} + U_R \xi L = 0 \] \hspace{1cm} (3.9)

The solutions of the two equations are

\[ U_T = \frac{e^{i\xi L} U}{e^{-i\xi L} - e^{i\xi L}} \]  \hspace{1cm} (3.10)

and

\[ U_R = \frac{-e^{i\xi L} U}{e^{-i\xi L} - e^{i\xi L}} \] \hspace{1cm} (3.11)

and thus Equation 3.7 can be expressed in the form

\[ u_{zp} = -\frac{U \sin[\xi(z - L)]}{\sin(\xi L)} e^{-i\omega t} \] \hspace{1cm} (3.12)

The approximation of axial displacement at a given radial coordinate also exhibits the phenomenon of resonance. The amplitude of \( u_{zp} \) is infinite as \( \sin(\xi L) \) approaches to zero. That is,

\[ \xi L = n\pi \] \hspace{1cm} (3.13)

where \( n \) is an integer. Because of the relation

\[ \xi = \frac{2\pi}{\lambda} \] \hspace{1cm} (3.14)
where \( \lambda \) is the wavelength, equation 3.13 can be written as

\[
L = \frac{n\lambda}{2}
\]  

(3.15)

where the integral number, \( n \), is termed the harmonic number. The resonance of a pile occurs when the pile length, \( L \), equals an integral number of half wavelength.

The resonant frequency is used for estimating the phase velocity. Resonant frequency of a pile under test can be excited in a broad frequency bands by selected input vibration. Each resonant peak, \( f \), is corresponding to a harmonic number, \( n \). When the pile length is known, the phase velocity can be derived in the manner.

\[
c = f \cdot \lambda
\]  

(3.16)

An example of using the resonant method to measure phase velocity is given in Figure 3.3 and Table 3.3. The resonant peaks are measured from the frequency response, \( \frac{V(\omega)}{F(\omega)} \), of an impulse response result. Use the nominal pile length found from the construction record, the phase velocity of the propagation wave can be calculated by equations 3.15 and 3.16. Satisfactory agreement is observed between the experimentally-computed phase velocities and the numerically-derived phase velocities from the numerical model based on shear modulus ratio of 325 and Poisson's ratio of 0.21 as shown in Figure 3.4.

The resonant method gives a good approximation for the measurement of phase velocity of individual wave mode. However, the measured resonant frequencies sometimes represent superimposed resonant peaks from multiple reflection sources, and
Table 3.3: An Example of Phase Velocity Computation Using the Measured Resonant Frequency: Test Data: Shaft 2, NGES, Amherst, MA; Nominal Shaft Length: 15.14 m, Poisson’s Ratio, $\nu = 0.21$, Shear Modulus Ratio=325, Equivalent Diameter=0.96m

<table>
<thead>
<tr>
<th>Harmonic Number</th>
<th>Resonant Peak $f$, (Hz)</th>
<th>Wavelength $\lambda(m) = \frac{2L}{n}$</th>
<th>Phase Velocity $c(m/s) = f\lambda$</th>
<th>3D Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>137</td>
<td>30.3</td>
<td>4140</td>
<td>3980</td>
</tr>
<tr>
<td>2</td>
<td>264</td>
<td>15.1</td>
<td>3992</td>
<td>3954</td>
</tr>
<tr>
<td>3</td>
<td>400</td>
<td>10.1</td>
<td>4041</td>
<td>3943</td>
</tr>
<tr>
<td>4</td>
<td>518</td>
<td>7.6</td>
<td>3918</td>
<td>3933</td>
</tr>
<tr>
<td>5</td>
<td>664</td>
<td>6.1</td>
<td>3844</td>
<td>3918</td>
</tr>
<tr>
<td>6</td>
<td>820</td>
<td>5.1</td>
<td>3992</td>
<td>3908</td>
</tr>
<tr>
<td>7</td>
<td>977</td>
<td>4.0</td>
<td>3917</td>
<td>3918</td>
</tr>
<tr>
<td>8</td>
<td>1074</td>
<td>3.8</td>
<td>3778</td>
<td>3874</td>
</tr>
<tr>
<td>9</td>
<td>1162</td>
<td>3.4</td>
<td>3815</td>
<td>3846</td>
</tr>
<tr>
<td>10</td>
<td>1260</td>
<td>3.0</td>
<td>3683</td>
<td>3812</td>
</tr>
</tbody>
</table>
Figure 3.3: An Example of Phase Velocity Computation Using the Measured Resonant Frequency: Data from the Frequency Response of the Impulse Response Test Conducted on Shaft 2, NGES, Amherst, MA
Figure 3.4: Numerically and Experimentally (Resonant Method) obtained Phase Velocities: Experimental Data from the Frequency Response of the Impulse Response Test Conducted on Shaft 2, NGES, Amherst, MA; Numerical data: Shaft length=15.14m, Equivalent diameter=0.96m, Poisson's ratio, $\nu = 0.21$, Shear Modulus Ratio, $\frac{G}{G_s} = 325$
not just reflections from the bottom of the pile. This reality may lead to errors in calculated group velocity. Nevertheless, for \( L(0,1) \) waves at low frequency, it is usually assumed that the all resonant intervals are the same regardless the number of reflection sources. This assumption is considered reasonable because the impedance ratio between pile toe and soil in the bottom is usually much larger than impedance ratio corresponding to other sources of reflection. Furthermore, the phase wave is not dispersive within this range based on theoretical evaluation. Therefore, the average phase velocity can be calculated by

\[
c = 2\Delta f L
\]

(3.17)

where \( \Delta f \) represents the average of resonant peak interval.

This example demonstrates the case where only one wave mode propagates at a given frequency. However, two or more wave modes may be excited when conducting a guided wave test at high frequencies. Thus, one must be able to identify resonant peaks and corresponding harmonic numbers when conducting high frequency tests. The use of the algorithm of Joint Time-Frequency Analysis (JTFA) provide a feasible solution for multiple mode analysis.

### 3.2.3 Geometrical Attenuation

As summarized in section 2.6, the imaginary part of wave number in the dispersion relation quantitatively represents the geometric attenuation of a propagating wave in
an embedded pile. The attenuation coefficient, $\vartheta$, is expressed in nepers as

$$\vartheta = \log_e \left( \frac{u_2}{u_1} \right) \text{ nepers}$$  \hspace{1cm} (3.18)

where $u_1$ and $u_2$ are the ratio of the displacement amplitude of the signals at the initial and final measuring positions, respectively. According to the derivation in section 3.2.2, the displacement $u$ for the longitudinal mode can also be represented as

$$u = u(r)e^{i(\omega t-\xi z)}e^{\xi iz}$$  \hspace{1cm} (3.19)

where $\xi_r$ is the real part of the wave number and $\xi_i$ is the imaginary part of the wave number. By considering a fixed point along the radius, $u(r)$ will be a constant. The second term represents a wave travelling in the axial direction, $z$, with phase velocity, $c = \frac{\omega}{\xi_r}$. The third term depicts the travelling wave decay exponentially in the axial direction, $z$.

Substituting equation 3.19 into equation 3.18, the attenuation coefficient, $\vartheta$, can be expressed as

$$\vartheta = \xi_i (z_2 - z_1)$$  \hspace{1cm} (3.20)

Because $z_1 - z_2$ are the length between initial and final measurement positions, the attenuation coefficient, $\vartheta$, can further be expressed in terms of nepers per unit length as

$$\vartheta = \xi_i \text{ nepers/length}$$  \hspace{1cm} (3.21)

$$\vartheta = 8.686\xi_i \text{ dB/length}$$  \hspace{1cm} (3.22)
The following example illustrates using the attenuation dispersion relation to determine the frequencies at which certain guided waves would propagate and be discernible to a well-designed measurement system. The assumed material properties and pile dimensions used in the example are listed in Table 3.4.

Table 3.4: The Assumed Parameters for the Illustrative Example of Using Attenuation Dispersion Curves

<table>
<thead>
<tr>
<th>Soil—Pile Parameters</th>
<th>Pile Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density Ratio, $\frac{\rho_p}{\rho_s}$</td>
<td>1.3</td>
</tr>
<tr>
<td>Shear Modulus Ratio, $\frac{\mu_p}{\mu_s}$</td>
<td>325</td>
</tr>
<tr>
<td>Poisson’s Ratio of the Soil, $\nu_s$</td>
<td>0.3</td>
</tr>
<tr>
<td>Poisson’s Ratio of the concrete $\nu_p$</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>Length (m)</td>
</tr>
<tr>
<td></td>
<td>Diameter (m)</td>
</tr>
</tbody>
</table>

Assuming the attenuation of -40 dB (1%) of the original amplitude is the limit after which signals are not discernible, the critical non-dimensional attenuation, $(\xi_a)_c$, for the conditions given in Table 3.4 can be determined by the following calculation: The maximum allowable attenuation for round trip signals, $(\xi_i)_c \cdot 2L = -40\text{ dB} = -4.605\text{ neper}$. Therefore, $(\xi_i)_c = \frac{-4.605}{2 \cdot 6.1} = -0.337(\text{neper/m})$. Because $a=0.153\text{ m}$, the non-dimensional critical attenuation coefficient $(\xi_a)_c=-0.057\text{ neper}$.

The attenuation dispersion curves for the above example were numerically derived by the symbolic program developed by Finno et al. (2002) using the software package Waterloo Maple. As illustrated in Figure 3.5, the passbands of the corresponding group waves can be identified by mapping the portion of the frequency band
Table 3.5: The Selected Frequency Range for Guided Wave Test Based on the Proposed Example: radius, $a = 0.5'$, bulk shear velocity, $c_T=2400\text{m/s}$

<table>
<thead>
<tr>
<th>Mode</th>
<th>Non-dimensional Frequency Range $(\Omega = \frac{\omega a}{c_T})$</th>
<th>Frequency Range (kHz) $(f = \frac{\omega}{2\pi})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L(0, 1)$</td>
<td>0 – 1</td>
<td>0–2500</td>
</tr>
<tr>
<td>$L(0, 2)$</td>
<td>4.5 – 5.5</td>
<td>11200–13800</td>
</tr>
<tr>
<td>$L(0, 3)$</td>
<td>8.2 – 8.8</td>
<td>20600–22100</td>
</tr>
</tbody>
</table>

of the attenuation dispersion curves where the non-dimensional wave numbers $(\xi a)$ are greater than the estimated $(\xi a)_c$, -0.057. The waves within the frequency bands corresponding to dispersion coefficients greater than -0.057 are considered capable of travelling down to the toe and reflecting back to the top of the example pile.

The passband of the group wave dispersion curves in Figure 3.5 indicates that for $L(0, 1)$, $L(0, 2)$ and $L(0, 3)$ modes, only one mode propagates at a given frequency. Of these possible waves, the ones with approximately constant velocities within the passband are listed in Table 3.5. These waves would be possible candidates for use on a guided wave experiment on the example pile.

In summary, the effects of dispersive attenuation and pile length cause reflected waves from the bottom of a pile to be detected only in certain frequency ranges. These frequency ranges are considered as the passbands of the propagation wave. Waves with frequency components outside the passband are not expected to be detected because all energy dissipates before it can reflect back the top of pile. This
Figure 3.5: An Example of the Filter Effect of the Surrounding Soils, $\frac{\mu_p}{\mu_s} = 325$, $\frac{\rho_p}{\rho_s} = 1.3$, $\nu_p = 0.23$: (a) Geometric Attenuation, $\xi_i a_i$, (b) The Passbands of the Group Wave Dispersion Curves for $L(0,1)$ to $L(0,5)$
characteristic may be limited in its usefulness when an unexpected reflection occurs at a shallow depth, and hence the passband range would be different since the travel path would be different than assumed in such a calculation.

### 3.2.4 Mode Shape

The mechanical impedance, \( Z \), for a longitudinal wave in a cylindrical pile is by definition,

\[
Z = \rho c A
\]  

(3.23)

where \( \rho \), \( c \) and \( A \) are concrete density, bulk wave velocity, and cross sectional area, respectively. When a propagation wave confronts a location where the impedance changes, part of the wave will reflect back and part of it will transmitted through. Figure 3.6 shows examples of typical defects that change the impedance of the waveguide. Poor quality concrete has lower of bulk wave velocity and density than good quality concrete. Necking and caving cause reductions in cross sectional area. Both conditions induce smaller mechanical impedance such that the ratio of the impedance, \( K \), between the concretes of the transmitted wave and the incident wave will be less than one. Fundamental one-dimensional wave theory (Bedford and Drumheller, 1994) indicates the reflection wave, \( R(t, z) \), can be computed by multiplying the incident wave \( I(t, z) \) by a wave reflection factor, \( \frac{1 - K}{1 + K} \) using:

\[
R(t, z) = \frac{1 - K}{1 + K} \cdot I(t, z)
\]  

(3.24)

Because the impedance ratio \( K \) is less than one, the value of the wave reflection factor is greater than zero which implies the reflection wave has the same phase
Figure 3.6: (a) Poor Quality of Concrete. (b) Necking (c) Caving

as the incident wave. However, interpretation based on impedance change is not sufficient to identify type of defects. For example, the defects illustrated in Figure 3.6 induce the same impedance value for an uniformly distributed propagation wave. All of these defects could be interpreted as the same type of anomaly: reduction of cross sectional area. For waves with non-uniform displacement profiles, an incident wave mode may not be able to sense the change of impedance caused by a defect, whereas another wave mode may be able to sense this same defect. This characteristic of modal shape can potentially be used to identify defects. The optimal measurement location for wave mode can be evaluated by measuring the vibration simultaneously along the radius of a pile at multiple locations.

To illustrate this idea, the three modes, $L(0,1)a$, $L(0,2)a$ and $L(0,3)a$ listed in Table 3.6 are selected based on the frequency range in Table 3.5. These modes propagate at approximately the same group velocity and attenuation but with different
mode shape. The displacement profile of the these modes are plotted in Figure 3.7.

Mode L(0,1)a has uniformly distributed displacement profile. It represents a plane wave propagating along a shaft that will reflect from all types of sufficiently large defect. Figure 3.7(b) shows that the displacement distributions of mode L(0, 2)a are concentrated at the outer edge of the pile. Consequently, a defect located near the pile-soil interface, such as necking, has more chance to be detected than a defect in the center of the pile when this L(0, 2) mode is induced in the pile. Figure 3.7(c) shows the displacements of the L(0, 3)a mode are concentrated near the center and edge of the pile. Defects in the center area and soil-pile interface can both be detected. Because the magnitude of the displacement at different distances along the radius is different for L(0, 2)a and L(0, 3)a, the optimal measurement locations are mode dependent. Intuitively, the locations at which the displacement is maximum is considered as the optimal location for vibration measurement and mode identification. Table 3.7 summaries the modes and the corresponding defects that can be identified for the example pile.
Figure 3.7: The Displacement Modal Shape for L(0,1)a, L(0,2)a and L(0,3)a
Table 3.7: An Example of Using Modal Shape to Identify Different Defects

<table>
<thead>
<tr>
<th>Type of Defect</th>
<th>$L(0, 1)a$</th>
<th>$L(0, 2)a$</th>
<th>$L(0, 3)$</th>
<th>Identifiable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Necking</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
<td></td>
</tr>
<tr>
<td>Caving</td>
<td>Y</td>
<td>N</td>
<td>Y</td>
<td></td>
</tr>
</tbody>
</table>

3.2.5 Universal Mode

The universal mode is herein defined as the mode at a dispersion curve that is independent of the Poisson’s ratio. The concept of universal points for traction-free infinite solid cylinders was proposed by Hudson (1943) who pointed out a universal point for the first symmetric branch at $\Omega = \sqrt{2}\xi a$, where $\xi a$ is the first non-zero root of $J_1''[(\xi a)] = 0$. Zemanek (1971) further indicated that there is an infinity number of these universal points, one for each branch of the symmetric mode. They are given by

$$\Omega = \sqrt{2}\xi a$$ (3.25)

where $\xi a$ is the solution of the derivation of the Bessel Function of the first kind

$$J_1''[(\xi a)] = 0$$ (3.26)

The first few roots of Equation 3.26 multiplied by $\sqrt{2}$ are 2.6036, 7.5392, 12.0576 and 16.5546. These values individually represent the non-dimensional frequency of the universal mode for $L(0,1)$, $L(0,2)$, $L(0,3)$ and $L(0,4)$ branches, respectively.
For a pile embedded in soil, extra boundary conditions make the wave propagation problem more complex. Nevertheless, numerical evaluation conducted by Hu (1999) indicated that the real part of the dispersion relation is independent of the shear modulus ratio and density ratio of the pile-soil system, but is a function Poisson's ratio, \( \nu \), of the pile. Finno et al. (2001) interpreted additional impulse response tests in light of the results of guided wave theory. Additional analysis conducted as part of this research identified the existence of a universal mode for each real and imaginary longitudinal branch at the same non-dimensional frequency. The non-dimensional frequency of the universal mode is independent of the Poisson's ratio, shear modulus ratio, and density ratio such that it is a constant for all pile-soil systems. These modes are called "universal" herein, and impact the experimental design because of their special characteristics in the dispersion relations and modal shapes.

Figure 3.8 shows two modes, \( L1a \) and \( L1b \), independent of the Poisson's ratio on the \( L(0,1) \) branch for the pile-soil system with \( \frac{\mu_p}{\mu_s} = 325 \) and \( \frac{\rho_p}{\rho_s} = 1.3 \). The \( L1a \) mode at the frequency \( \Omega = 2.60 \) can be observed on the attenuation, phase velocity, and group velocity dispersion curves. However, the \( L1b \) mode at the frequency \( \Omega = 1.40 \) can only be identified on the attenuation and group velocity dispersion curves, and is absent from the phase velocity dispersion curves. Figure 3.9 shows \( L2a \) independent of the Poisson's ratio at \( \Omega = 7.5 \), can be identified for the \( L(0,2) \) branch on the group velocity, phase velocity, and attenuation dispersion curves. Figure 3.10 shows for \( L(0,3) \) branch, \( L3a \) and \( L3b \) modes independent of the Poisson's ratio can be identified at the non-dimensional frequencies of 12 and 7.3, respectively. Similar to the \( L(0,1) \) branch, mode \( L3a \) is found on all dispersion curves while mode \( L3b \)
is independent of Poisson's ratio only on the attenuation and group wave dispersion curves.

Table 3.8 compares the theoretically-derived value for traction-free cylinders with those numerically-derived for embedded pile-soil system. The comparison shows that the non-dimensional frequencies for $L1a$, $L2a$, and $L3a$ modes are consistent with the first three non-dimensional universal frequencies of the traction-free piles. The non-dimensional frequencies corresponding to $L1b$ and $L3b$, however, cannot be observed for traction-free piles. The $L1b$ and $L3b$ are also absent from the phase velocity dispersion curves which implies they are not the universal modes.
Figure 3.8: The modes independent of the Poisson's ratio, $L1a$ and $L1b$, at the $L(0,1)$ Branch: (a) Effect of Poisson's ratio on attenuation dispersion curve, (b) Effect of Poisson's ratio on group wave and phase wave dispersion curves where the soil-pile system is with the parameters: $\rho$-ratio=1.3, $\mu$-ratio=325, and $\nu_r=0.23$. 


Figure 3.9: The modes independent of the Poisson’s ratio, L2a, at the L(0,2) Branch: (a) Effect of Poisson’s ratio on attenuation dispersion curve, (b) Effect of Poisson’s ratio on group wave and phase wave dispersion curves where the soil-pile system is with the parameters: \( \rho \)-ratio=1.3, \( \mu \)-ratio=325, and \( \nu_p=0.23 \)
Figure 3.10: The modes independent of the Poisson’s ratio, $L3a$ and $L3b$, at the $L(0,3)$ Branch: (a) Effect of Poisson’s ratio on attenuation dispersion curve, (b) Effect of Poisson’s ratio on group wave and phase wave dispersion curves where the soil-pile system is with the parameters: $\rho$-ratio=1.3, $\mu$-ratio=325, and $\nu_p=0.23$
<table>
<thead>
<tr>
<th>Mode</th>
<th>Non-dimensional Frequency, $\Omega$</th>
<th>3-D Numerical Approach</th>
<th>Identified at the Dispersion Curve</th>
<th>Attenuation Group Velocity</th>
<th>Phase Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1a</td>
<td>2.60</td>
<td>2.6036</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>L1b</td>
<td>1.41</td>
<td>NA</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>L2a</td>
<td>7.5</td>
<td>7.5392</td>
<td>Y</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>L3a</td>
<td>12.5</td>
<td>12.0576</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
<tr>
<td>L3b</td>
<td>7.2</td>
<td>NA</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
</tr>
</tbody>
</table>
Evaluation of the modes independent of the Poisson's ratio is illustrated by mode L1a. The evaluation is focused on the evolution of power flux and displacement profiles for the modes adjacent to L1a. Properties that may be useful for practical guided wave tests are summarized.

Figure 3.11 shows the evolution of the power distribution of the modes adjacent to the L1a from $\Omega=2.55$ to $\Omega=2.70$ for values of Poisson's ratio of 0.14 to 0.28 computed by the programs developed by Hanifah (1999). These power profiles exhibit the same pattern that are concavely downward and having maximum power flux equal to one at $R \approx 0.80$. However, the mode shapes of the power flux corresponding to Poisson's ratio of 0.14 and 0.28 merge at frequencies close to L1a which implies the power distribution is independent of the Poisson's ratio at the L1a mode.

Figure 3.12 shows the evolution of the distribution of the real and imaginary parts of axial displacement for the modes adjacent to L1a from $\Omega=2.55$ to 2.70. As can be seen, the value of

$$|Re(u_z) - Im(u_z)|$$

(3.27)

at any location across the pile decreases with increasing frequency and approaches to zero at the frequency corresponding to L1a mode, and increases as $\Omega > 2.60$. In this range of $\Omega$, the influence of the Poisson's ratio on $|Re(u_z) - Im(u_z)|$, is insignificant.

The steady-state solution for the axial displacement, $u_{zp}$, can be expressed by Equation 3.5

$$u = U e^{i\omega t}$$
Figure 3.11: The evolution of power flux distribution as a function of $\Omega$ for modes of the $L(0,1)$ branch adjacent to the universal mode $L1a$
where $U$ is an amplitude function of $\omega$ and $\xi$. For a fixed-end pile, Equation 3.5 can be expressed as Equation 3.12 which verifies the occurrence of end resonance. For the boundary condition that the pile tip is traction-free or a continuous interface adjacent to the bottom soil, the closed form solution of the $u_z$ may need to be modified.

### 3.2.6 Material Attenuation

Theoretical evaluation in Section 3.2.3 shows that the guided waves in an embedded drilled shaft can only propagate in the frequency range in which the geometrical attenuation coefficient is smaller than a critical value. As shown in Figure 3.5, the geometrical attenuation coefficient for all longitudinal modes at high frequency are asymptotic to a very small value. This trend implies that the effect of geometrical attenuation is not of concern for waves that propagate at very high frequency because material attenuation as a result of absorption and scattering will dominate the energy dissipation.

It is recognized that the energy loss as a result of absorption and scattering at the interfaces of aggregates prevent the wave from propagating when the wavelength of a propagation wave is smaller than the size of the aggregates in the concrete (Gaydeck, 1992). The wavelength, $\lambda$ is related to frequency by

$$f = \frac{c_p}{\lambda}$$  \hspace{1cm} (3.28)

where $c_p$ is phase wave velocity. When $\lambda$ is equal to the mean aggregate size, $d$, the corresponding frequency approximately will be the maximum frequency that a wave
Figure 3.12: The evolution of displacement distribution as a function of $\Omega$ for the modes adjacent to the universal mode $L1a$. 
can propagate with. Assuming the bulk shear wave velocity, $c_T$ of $2300 m/s$ in concrete, the maximum useful frequency for guided wave propagation can be estimated from Equation 3.28 based on the assumption that

$$c_p = c_T$$

Table 3.9 summarizes the maximum useful frequencies for concrete cylinders with diameters ranging between 15 cm and 46 cm and aggregate sizes from 1 cm to 7.6 cm. As can be seen, the maximum useful frequency limited by the effect of wave scattering and absorption along the cement-aggregate interface depends on mean aggregate size. The corresponding maximum useful non-dimensional frequency can be calculated by

$$\Omega = \frac{2\pi f a}{c_T}$$  \hspace{1cm} (3.29)$$

where $f$ is the maximum useful frequency, $a$ is the pile radius and $c_T$ is the bulk shear wave velocity. The maximum useful non-dimensional frequency is a function of the pile diameter.

### 3.3 Summary

The objective of this Chapter is to establish the information database that is necessary for building up the experimental system and conducting the non-destructive guided wave test based on the numerically-derived 3-D guided wave theory.
Table 3.9: The Maximum Useful Frequencies for Concrete Pile Tests Based on the Consideration of the Material Damping

<table>
<thead>
<tr>
<th>Mean Aggregate Size</th>
<th>Pile Diameter</th>
<th>(^{a}) Maximum Useful Frequency, (f_{max}) (kHz)</th>
<th>(^{b}) Maximum Useful Non-dimensional Frequency ((\Omega_{max}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1((\frac{3}{8}))</td>
<td>15.2 (6)</td>
<td>242.1</td>
<td>50.2</td>
</tr>
<tr>
<td>1((\frac{3}{8}))</td>
<td>20 (8)</td>
<td>242.1</td>
<td>66.1</td>
</tr>
<tr>
<td>1((\frac{3}{8}))</td>
<td>25.4 (10)</td>
<td>242.1</td>
<td>84</td>
</tr>
<tr>
<td>1((\frac{3}{8}))</td>
<td>30.5 (12)</td>
<td>242.1</td>
<td>100</td>
</tr>
<tr>
<td>1((\frac{3}{8}))</td>
<td>45.7 (18)</td>
<td>242.1</td>
<td>151</td>
</tr>
<tr>
<td>2.5(1)</td>
<td>15.24 (6)</td>
<td>90.6</td>
<td>19</td>
</tr>
<tr>
<td>2.5(1)</td>
<td>20 (8)</td>
<td>90.6</td>
<td>25</td>
</tr>
<tr>
<td>2.5(1)</td>
<td>25.4 (10)</td>
<td>90.6</td>
<td>31</td>
</tr>
<tr>
<td>2.5(1)</td>
<td>30.48 (12)</td>
<td>90.6</td>
<td>38</td>
</tr>
<tr>
<td>2.5(1)</td>
<td>45.72 (18)</td>
<td>90.6</td>
<td>57</td>
</tr>
<tr>
<td>5.1(2)</td>
<td>15.24 (6)</td>
<td>44.7</td>
<td>9.4</td>
</tr>
<tr>
<td>5.1(2)</td>
<td>20 (8)</td>
<td>44.7</td>
<td>12.3</td>
</tr>
<tr>
<td>5.1(2)</td>
<td>25.4 (10)</td>
<td>44.7</td>
<td>15.6</td>
</tr>
<tr>
<td>5.1(2)</td>
<td>30.5 (12)</td>
<td>44.7</td>
<td>18.8</td>
</tr>
<tr>
<td>5.1(2)</td>
<td>45.7 (18)</td>
<td>44.7</td>
<td>28.2</td>
</tr>
<tr>
<td>7.6(3)</td>
<td>15.24 (6)</td>
<td>30.2</td>
<td>6.3</td>
</tr>
<tr>
<td>7.6(3)</td>
<td>20 (8)</td>
<td>30.2</td>
<td>8.3</td>
</tr>
<tr>
<td>7.6(3)</td>
<td>25.4 (10)</td>
<td>30.2</td>
<td>10.5</td>
</tr>
<tr>
<td>7.6(3)</td>
<td>30.5 (12)</td>
<td>30.2</td>
<td>12.6</td>
</tr>
<tr>
<td>7.6(3)</td>
<td>45.7 (18)</td>
<td>30.2</td>
<td>18.9</td>
</tr>
</tbody>
</table>

\(^{a}\) Based on the consideration of wave scattering and refracting at the cement-aggregate interface.
\(^{b}\) Normalized by Equation 3.29
Section 3.2.1 illustrates the approach used to identify if two or more modes of propagation waves can be excited at a given frequency based on the real part of the dispersion relations for traction-free or embedded piles.

Section 3.2.2 demonstrates the approach for measuring the group velocity and phase velocity. The group velocity can be measured directly from the recorded time waveform because the wave energy is carried by each group of wave pocket. The phase wave is enveloped by the group waves and cannot be measured directly from the time history of the response. Nevertheless, it can be estimated from the resonant spectrum by

\[ c = f\lambda \]

based on the principle that an integer number of half wavelength equals to the pile length.

Section 3.2.3 illustrates the approach to quantitatively evaluate the effect of geometrical attenuation on wave propagation. The selection of the frequency band within which the wave mode has favorable properties for integrity evaluation is also presented.

Section 3.2.4 presents the concepts of integrity evaluation based on the power or displacement distribution of a selected wave mode.

Section 3.2.5 introduces the universal mode which is independent of the shear modulus ratio, density ratio, and Poisson's ratio of the pile. If a resonance corre-
sponding to the universal mode can be excited and identified in practical application, the shear wave velocity can be determined without having to know Poisson's ratio. This determination is important for realistic data analysis because the measured data can be normalized by the shear wave velocity and compared with the numerical results.

Section 3.2.6 presents the method for estimating the upper frequency limits for conducting guided wave test on concrete material. This limit is caused by material attenuation which is a result of wave scattering and absorption at the interface between the cement and aggregate particles.
Chapter 4

Test System

4.1 Overview

Chapter 2 and 3 show that the longitudinal modes, propagation velocities, attenuation and mode shape at given frequencies can be predicted by results of numerical analyses. They also identify preferred frequency ranges for conducting guided wave test. This chapter describes the instrumentation and equipment that comprise the test system. The test apparatus is composed of three parts: a vibration generation system, a vibration measurement assembly, and a PC-based control system. The vibration sources include both modal hammers and a modal shaker. A modal hammer, although inefficient in frequency content control, is used to generate stress waves with low frequency components. The modal shaker, a piezoelectric stress wave source, is used to generate waves with higher frequency components. The frequency content of the waves can be controlled by the modal shaker. The excited vibrations are measured by charge-mode accelerometers that have wide and flat frequency responses. The measured vibrations are then sent back to the PC-based control system, consisting of a portable personal computer with one or more data acquisition boards and the
control program developed with LabView software. The PC-based control system functions as both a pulse generator and an oscilloscope. The vibration shaker is controlled by waveform signals modulated by a virtual pulse generator and power amplifier. The accelerometers are powered by an external signal conditioner. Multi-channel data acquisition is controlled by a virtual oscilloscope. The embedded analog and digital convertor (A/D converter) in the DAQ board digitally converts the analog voltage signals from channel to channel through the embedded multiplexer. Data averaging and spectrum analyses are programmed as part of the virtual oscilloscope program so that the preliminary evaluation of the test results can be done in real time as soon as the data have been acquired.

4.2 Vibration Generator and Transducer

4.2.1 Vibration Generator - Modal Hammer

A modal hammer is used to generate transient vibrations up to several kilohertz for lower frequency vibration tests. Two modal hammers are used for this study, models PCB-086M54 and PCB-086C04 manufactured by PCB Piezoelectronics. The former is used for testing full-sized drilled shafts while the latter is used for prototype pile experiments. The required frequency range is application dependent. For smaller diameter piles, higher frequency response is preferred because the frequency at which dispersion occurs is higher than for larger diameter piles. The head of the 086M54 hammer tip is 5.1 cm (2 in) in diameter and can induce a force with a magnitude as high as 22 kN (5000 lb). Properly struck, the modally tuned hammer eliminates multiple impacts. The force is recorded with an embedded transducer mounted on
the hammer head. This force transducer develops an electrical charge proportional to applied pressure. The sensitivity of the hammer is 0.20 mV/N (0.90 mV/lb). The magnitude of the force created by a hammer impact is calculated by dividing the measured voltage by the sensitivity.

This hammer has been used in a number of studies concerning impulse response evaluations of drilled shafts (e.g., Finno and Gassman 1998; Gassman and Finno 1999). Satisfactory results were reported for pile depths up to 27.48 m and length to diameter \( \frac{L}{D} \) ratio as high as 30. The head of the PCB 086C04 hammer tip is 0.63 cm (0.25 in) in diameter and can induce a force with a magnitude as high as 4.4 kN (1000 lb). The modal tuned mechanism keep the hammer from generating multiple impacts. The sensitivity of the embedded force gage is 1.19 mV/N (5.28 mV/lb). This hammer is capable of inducing much higher frequencies than PCB-085M54 hammer. However, the application on large scale foundations is limited because the energy being excited is relatively small.

Several interchangeable tips of varying stiffness are provided for both PCB-086M54 and 086C04 hammers. Different tip stiffness creates various impact durations and allows the user to control the input frequency range by changing the hammer or hammer tip. Frequency content is essentially a function of contact time and is significantly affected by the stiffness of hammer tip. Other factors such as hammer mass, impact velocity and spring constant of the impact surface also play important roles on the induced frequency range. The actual useful frequency range being excited in an impact is determined from the frequency distribution transformed from the force-time
function of an elastic impact (Carino and Sansalone, 1986). Generally, the maximum useful frequency, $f_{\text{max}}$, for impulse response test on pile foundation is

$$f_{\text{max}} = \frac{1}{\Delta t} \quad (4.1)$$

where $\Delta t$ is the duration of hammer impact.

Wave generated by elastic hammer impact are usually composed of lower frequency components and distributed over a broad frequency range starting near 0 frequency. Theoretical evaluation (Finno et al., 2001) shows that waves propagate at a constant velocity over the frequency range excited by hammer impact when testing relative small diameter piles. The wave propagation behavior changes drastically at higher frequencies at which the propagation velocity becomes frequency dependent. At these higher frequencies, the waves will propagate within certain frequency ranges and evanesce at other frequency ranges. More than one mode of waves may propagate at a given frequency. To better evaluate these characteristics, the equipment was developed to create waves with narrow-band frequency components, other than broad-band waves. A modal shaker, which is able to create continuous or transient waveforms at controlled frequencies, is used as a vibration source for narrow band waveform generation.

### 4.2.2 Vibration Generator - Modal Shaker

A commercially available vibration shaker, model number WR-F7 manufactured by Wilcoxon Research Inc, is used as the specific vibration source for the higher fre-
Table 4.1: Blocked Force Output of WR-F7 at 800 V rms Input

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Force Peak (lb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 to 40 kHz</td>
<td>8</td>
</tr>
<tr>
<td>1.8 to 30 kHz</td>
<td>40</td>
</tr>
<tr>
<td>2.5 to 20 kHz</td>
<td>100</td>
</tr>
</tbody>
</table>

quency applications. This shaker utilizes the expansion and contraction properties of piezoelectric crystals for sonic and untrasonic structural excitation. It is portable, light weight (2.5 lb) and can be stud mounted directly to a structural member without external support. A transducer base, model number of WR-Z7, is embedded in the head of the generator as the interface between the generator and a specimen. The transducer base contains a force gage and an accelerometer to measure the force applied to the structure and the resulting acceleration.

This vibration generator can be operated at frequencies from 500 Hz to over 60 kHz. However, wide fluctuation in force output are present at high frequencies and accurate control of the frequency may not be possible in all cases. Table 4.1 lists the manufacturer-provided range of the force output measured at the impedance head, or the block force output as the shaker is mounted on an object with infinite mechanical impedance. The voltage sensitivity of the force gage is 22.9 mV/N (100.8 mV/lb) and that of the accelerometers is 11.6 mV/g. According to the manufacturer’s specification, the accelerometer has a ±3 dB flat frequency response between 10 Hz and 25 kHz.
To ensure better performance, the piezoelectric shaker must be driven by high voltage power supply with flat total harmonic distortion (THD) over a wide frequency range. For instance, the shaker requires 1300 Watts of output power to obtain 800 V rms at 20 kHz for the 0.015 mF load of the WR-F7 shaker. The impedance that WR-F7 presents to the power amplifier is capacitive, and therefore decreases with increasing frequency. Thus the resultant load on the amplifier increases with frequency. To improve amplifier performance so that the shaker can be driven at its full voltage level and to obtain a reasonably efficient energy transfer between the power amplifier and piezoshaker, proper impedance matching network is needed between these two components.

The power amplifier and matching network are manufactured by Wilcoxon Research and are model numbers WR-PA8C and WR-N8L, respectively. These instruments drive the WR-F7 piezoelectric shaker. The WR-PA8C model is a single-channel output, power amplifier that can be operated, according to specifications provided by manufacturer, with 1000 Watt continuous average output power into 4 ohms over the frequency range from 5 Hz to 40 kHz with no more than 0.07% THD. The WR-N7 matching network, with built-in 5 step-up transformer and selectable output voltages, is used for optimizing the impedance match between the power amplifier and the piezoelectric shaker at different frequencies. Table 4.2 lists manufactural recommended voltage selections and their corresponding step up ratios for several frequency ranges. The front pannel of the WR-N8L is shown in Figure 4.1 where a switch tap located on the left side has 5 selectable voltages. To obtain greater output from the shaker, the switch tap should be adjusted to higher voltage. However, as the fre-
Table 4.2: N8L Operation Setup

<table>
<thead>
<tr>
<th>Output Range Switch Position</th>
<th>Step Up</th>
<th>Frequency Range (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>60V</td>
<td>1:1</td>
<td>0.5 to &gt; 40</td>
</tr>
<tr>
<td>150V</td>
<td>2.5:1</td>
<td>1.5 to &gt; 40</td>
</tr>
<tr>
<td>300V</td>
<td>5:1</td>
<td>1.5 to &gt; 40</td>
</tr>
<tr>
<td>450V</td>
<td>7.5:1</td>
<td>1.5 to &gt; 40</td>
</tr>
<tr>
<td>800V</td>
<td>13.3:1</td>
<td>1.5 to &gt; 31</td>
</tr>
</tbody>
</table>

The frequency of the operation is increased, it is necessary to reduce the step up ratio for better impedance match.

Figure 4.1: Schematic of the Front Panel of WR-N8L Matching Network

The set-up of the vibration generation system is schematically shown in Figure 4.2. The vibration shaker WR-F7 is controlled by the signals amplified and transmitted from the PA8C power amplifier. Matching network N8L provides the impedance
matching between shaker and amplifier for better performance. The force applied to the structure and the resulting acceleration can be measured by the embedded Z7 transducer.

Figure 4.2: Schematic of the Vibration Generation System with WR-F7 Modal Shaker

Figure 4.3 shows the method of mounting the WR-F7 vibration shaker on a concrete surface. First, a smooth and flat area at least the size of the shaker base (1.5 inch in diameter) is ground on the mounting surface. Then, a hole with a diameter slightly greater than the 3/8-16 mounting stud is drilled to ensure the hole is perpendicular to the mounting surface. The mounting stud is epoxied vertically into the hole to the depth that the transducer base can rest flush against the concrete surface to maintain transducer sensitivity. This mounting method takes about 10 minutes, and, at this rate, tests can be conducted from pile to pile.
Figure 4.3: Schematic of the stud-mounting method for the WR-F7 Vibration Shaker
4.2.3 Accelerometer

Vibration response in the guided wave test is measured by accelerometers, model number PCB-W352A78 manufactured by Wilcoxon Research, Inc. Corresponding calibration data and key specifications are listed in Table 4.3.

Table 4.3: Calibration Data and Key Specifications of the PCB-W352A78 Accelerometer

<table>
<thead>
<tr>
<th>Serial No.</th>
<th>21934</th>
<th>21935</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calibration Data</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Voltage Sensitivity, mV/g (mV/m/s(^{-2}))</td>
<td>100.5 (10.3)</td>
<td>100.4 (10.2)</td>
</tr>
<tr>
<td>Resonant Frequency, kHz</td>
<td>49</td>
<td>51.5</td>
</tr>
<tr>
<td>Output Bias Level, Volt</td>
<td>10.2</td>
<td>10.0</td>
</tr>
<tr>
<td>Deviation of Frequency response at 15kHz</td>
<td>1.8 %</td>
<td>4.6 %</td>
</tr>
<tr>
<td>Key Specifications</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Range, ±g</td>
<td></td>
<td>50</td>
</tr>
<tr>
<td>Resolution, g</td>
<td></td>
<td>0.001</td>
</tr>
</tbody>
</table>

A line powered 8-channel signal conditioner, PCB-482A20, is used as the power supply for properly operating the accelerometers which require excitation voltage between 24 and 28 VDC and constant current between 2 and 20 mA. Selectable gains, \( \times 1, \times 10 \) and \( \times 100 \), in this signal conditioner provide the option for signal amplification. The power excitation and signal transmission of the assembly is schematically shown in Figure 4.4.
Figure 4.4: Schematic of the power excitation and signal transmission for accelerometer-signal conditioner assembly
The accelerometers can be adhesively mounted to the concrete surface using high grade grease. This direct mounting method is quick, economic and would not damage the transducer. However, the maximum useful frequency response of the assembly is only in the order of 1000 Hz. For higher frequency application, a much stiffer adhesive is needed to increase the useful frequency range. To increase this frequency range, an adhesive mounting base affixed to the concrete surface with epoxy as shown in Figure 4.5. This arrangement balances cost and frequency range with the need to easily mount and reuse the accelerometer. The accelerometer can be stud mounted on the adhesive mounting base quickly. Removing and reinstalling has proved to be easy and can be done without the risk of the accelerometer being damaged. The use of the mounting base eliminates the adhesive from being in direct contact with the accelerometer. The size of the mounting base ideally should be the same as the accelerometer. If such match is unavailable, the next larger-sized mounting base should be used. The guided wave test uses model number PCB-080A12 mounting bases manufactured by PCB Piezoelectronics, Inc., with hex size of $\frac{3}{4}$ inch and thickness of 0.2 inch to mount the PCB-W352A78 accelerometer with a base size of 0.65 inch.

To mount this assembly, first grind the concrete surface smooth and flat. Then, place a small portion of epoxy on the grooved side of the mounting base and firmly press the assembly on to the concrete surface. The accelerometer is stud-mounted to the base after the epoxy has hardened. Petro wax is placed between the mounting base and the accelerometer base to ensure better contact. The mounting thread of the PCB-W352A78 accelerometer is 10-32 female. Recommended mounting torque is 10 to 20 in-lb (113-225 N-cm) by hand-tightening the mounting stud with a torque
wrench, one maintains consistent mounting torque.

![Diagram of Adhesive Mounting](image)

Figure 4.5: Schematic of Adhesive Mounting

### 4.3 Overview of PC-based Control Systems

Prior to the era of personal computer, laboratory test systems were usually, if needed, operated by programmable instrumentation. This instrumentation was controlled by expensive, single purpose controllers. A General Purpose Interface Bus (GPIB) was developed by Hewlett-Packard in 1960s and used as the integral communication port for these controllers. When the general purpose personal computer became popular, engineers began to control benchtop instruments using this solution. By 1983, GPIB hardware interface had become the dominant industrial standard (IEEE 488.2) in the market for communications between instruments and computers. One GPIB interface can control up to 15 devices. At that time, the software to control the instruments
was written with a text-based programming language such as BASIC.

Development of more capable operating systems and computer architecture in recent years provided more sophisticated platforms for instrumentation and data acquisition and thus the concept of a virtual instrument (VI) became a reality. The objective of virtual instrumentation is to use a general-purpose PC to mimic real instruments with their dedicated control and displays, but with the added versatility that comes with software (Johnson, 1997). A computer-based system can be assembled with the fundamental hardware and software components to design virtual instruments for specific applications. The hardware may be plug-in boards, external instruments, or a combination of both. A software interface can be designed to serve a specific application in a flexible and customized feature. Users only need to focus on the subsets of an instrument's full capability, which in itself can be very complex.

Virtual instrumentation became increasingly important when the instrument-on-a-card system such as VXI and DAQ boards were developed. The industrial standard VXI, first established in 1987 by the VXI consortium, is the abbreviation of VMEbus (Versa-Modular Eurocard) eXtensions for Instrumentations. A VXI instrument is a card that plugs into a chassis containing other VXIs which in turn connect to a VXI controller board plugged in the BUS of a personnel computer. A VXI mainframe can have up to 20 slots. An individual VXI instrument does not have a front panel and must be controlled by a computer. A software front panel offers the control interface and communication capabilities for a VXI system. The combination of plug-in modules, and high performance timing and communication capabilities of the VXI system
makes it versatility and more complex than a GPIB system. Another instrument-on-a-card system is the computer bus plug-in DAQ board which is directly installed on the PCI or AT bus of a personnel computer, and makes a versatile instrument. Expandibility of the PC-based system depends on the number of BUS slots available on the main PC board. Several DAQ boards can be used in parallel for the demands of higher performance. The cost of the computer bus plug-in board system is significantly lower than that of either the VXI or GPIB systems.

Figure 4.6 illustrates some options for building up a PC-based control system. The serial instrument for which data are communicated through the serial port, is the least expensive choice. However, this system usually provides features only for a single purpose application, and the data are processed at a very low rate (one bit at a time). It is usually not a solution for building a high speed data acquisition system. Generally, GPIB and VXI based control systems provide better performance than the plug-in DAQ boards, but are much more expensive than the DAQ system. Chapter 3, the experimental design consideration, implies that the basic requirement for the test would be a PC based data acquisition system which is able to perform both waveform generation and triggered data acquisition in the time domain. The output waveforms of interested are a transient pulse wave and the highest frequency waves of interested for concrete structure are on the order of 50 kHz. The optional complicate data analyses can be conducted after the real time test without sacrificing the overall performance of the system. Based on these considerations, a DAQ system comprised of a PC and one or more plug-in multiple function DAQ boards was selected as the mainframe for the guided wave test.
4.4 Building the Guided Wave Experimental Approach

4.4.1 Overview

The guided wave experimental approaches developed herein focus on vibration measurement and excitation of axisymmetric mode wave propagation in free and embedded concrete cylinders. The frequency range of interested is from DC to around 50 kHz. In these experiments, the concrete cylinder (the pile) is treated as a wave guide within a soil mass. The range of relative stiffness between the concrete cylinder and the surrounding soil is 45 to 325, in terms of shear modulus ratio. The Poisson's ratio of the concrete cylinder varies from 0.18 to 0.28. In the experiment, one induces vibrations to the concrete via a controlled (modal shaker) or semi-controlled (modal hammer) input and measures and interprets wave propagation velocity and
relative surface vibration magnitude at different locations by means of accelerometers.

The control system only needs to perform multi-channel triggered data acquisition when the modal hammer is used as the vibration source. All the signal sources, including the hammer and accelerometers are connected to input channels of the PC-based control system. Because the excited vibrations are transient, only a finite length of the waveforms is of interest. Thus, it would not be necessary to acquire data over a long period of time, filling up memory and disk space. Start of the data acquisition is controlled by an event called a trigger, which specifies conditions required to start the data acquisition process. Once this event is identified at the triggering channel, all other input channels start to acquire data. The acquired waveforms at each channel use the same time base.

The acquired waveforms are analog and need to be digitally converted by the data acquisition hardware. The duration of the interested waveform can be interpreted as the number of samples, \( n \), to acquire, and can be initially estimated by the sampling rate, \( f_s \). For example, when an identifiable travel time of a propagation wave, \( t_e \) along a drilled shaft is estimated to be 50 ms, and the sampling rate, \( f_s \), is 10 kSample/sec. The number of samples to be acquired is calculated as:

\[
  n = f_s \cdot t_e = 10,000 \cdot 50 \cdot 10^{-3} = 500
\]

(4.2)

In many instances, spectrum analysis is required and the time-domain waveforms need to be transformed into frequency domain by Fast Fourier Transform (FFT).
The FFT algorithm has to be performed on data points that are power of 2. If the number of samples is not a power of 2, a sequence of zero data points needs to be added to the trail of the raw time waveform to make the sample number equal to the closest power of 2. For example, 12 zero data points will be added to the example in Equation 4.2 when conducting a FFT. For consistence and simplicity, the number of data points to acquire should be set equal to a number that is of the power 2, where

\[ n = 2^x \]  \hspace{1cm} (4.3)

where \( x \) is an integer and \( 2^{x-1} < f_s \cdot t_e < 2^x \)

A fundamental rule of sampling is that the input signal must be sampled at a rate, \( f_s \), greater than twice the highest frequency component, \( f_a \) in the signal. This is known as the Shannon sampling theorem, and the critical sampling rate is called the Nyquist rate. Violating the Nyquist criterion is called under-sampling and will result in an alias frequency. The high frequency component not being sampled will be exhibited as an alias frequency within the sampling frequency range. The HP Technique Manual (Hewlett-Packard, 1986) suggests that for a data acquisition system with interpolation function, to accurately capture a vibration response, waveforms should be sampled at a frequency that is at least four times the highest frequency in the signal

\[ f_s = 4 \cdot f_a \]  \hspace{1cm} (4.4)

The suggested sampling frequency implies that the sampling rate should be at least 200 kHz for a typical guided wave test on a concrete cylinder.
The number of samples, \( n \), and the sampling rate, \( f_s \), will determine the frequency resolution for the acquired waveform spectrum. The frequency resolution, \( \Delta f \), is the frequency interval between two adjacent discrete data points in the spectrum and can be calculated as the ratio of \( f_s \) and \( n \).

\[
\Delta f = \frac{f_s}{n}
\]  
(4.5)

It should be noted that the resulting spectrum contains only \( \frac{n}{2} \) data points.

Start of data acquisition can be controlled by a triggering event, such as a specified voltage level or slope. The data points acquired before the trigger event are called pretrigger scans. Pretrigger scans are part of the total number of samples to be acquired. Analog signals acquired from the signal source have to be converted to digital signals (A/D conversion) for later processing. However, white gaussian noise resulting from the A/D conversion will adversely affect the signal resolution, SR:

\[
SR = \frac{V}{2^k}
\]  
(4.6)

where \( V \) is a bipolar or unipolar voltage bias and \( k \) is the resolution bit of the instrument. The noises are theoretically zero-mean random variables when a common ground reference is set for all input channels. Noises can be reduced by averaging a number of in-phase acquisitions because a recurrent waveform is acquired by the same trigger event so that each acquired waveform is in the same part of the signal cycle. Only noise will be cancelled out by averaging arrays of data.
Waveform generation control is required when the modal shaker is used as the vibration source. Different waveforms must be digitally edited by a controlling program and converted into analog signals by Analog to Digital Converter (ADC) in the I/O hardware. The converted analog signals are then amplified and sent over to the shaker as the control source.

4.4.2 I/O Hardware

A plug-in multiple function I/O DAQ board is used to control the NDE testing system. Key specifications for selecting appropriate I/O hardware are:

- Portable and rugged PC-based control system.
- At least 2 PCI or AT bus slots are required after configuration of the computer system.
- Timing control.
- Four analog input (AI) channels.
- One analog output (AO) channel.
- Analog to digital convertor.
- Single channel sampling rate is at least 500 kHz.

Portable PC

A portable PC manufactured by ACME Portable Machine, Inc. has been customized so that guided wave tests can be conducted. The computer is used for controlling
the DAQ system, acquiring, analyzing, storing, and presenting data. The PC is configured with an Intel Pentium III CPU running at a clock rate of 833 MHz and with 768 Mb RAM. The portable PC is equipped with 15 inch color LCD display, a 30 Gigabyte hard drive and a SCSI CD-RW. Four PCI-slots are available for plug-in DAQ boards after the PC system is configured. Its durable ABS plastic chassis and internal protection steel frame make it suitable for field use.

Plug-in DAQ Board

The PCI-MIO-16E-1 DAQ board, a product of National Instrument, is used as the I/O hardware for the proposed PC-based control system. Key specification of this plug-in multiple function DAQ board are shown in Table 4.4. This board satisfies all the specified requirements except for the aggregate sampling rate. When the sampling rate for each AI channel is set to 500 kHz, the aggregate sampling rate for 4 channel AI is 2 MHz, which is more than its 1.25 MHz capability. Nevertheless, this problem can be solved by using multiple DAQ boards connected in parallel to each other. One of the DAQ boards is configured as the primary instrument while the timebase of the secondary DAQ board can be synchronized over the embedded Real-time system integration (RTSI) bus. The polarity and voltage range are software selectable.

Figure 4.7 schematically shows the configuration of the DAQ board as it is plugged into a PCI bus and connected to an external I/O connector. The input ground reference is set to referenced single ended (RSE) mode in which the floating analog signals are connecting to the positive input of the programmable gain instrumentation amplifier (PGIA) with a single ended connector. The negative input of the PGIA is
Table 4.4: Key Specifications of NI-PCI-MIO-16E-1 DAQ Board

<table>
<thead>
<tr>
<th>Specification</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of AI Channels</td>
<td>16 Single-end or 8 differential</td>
</tr>
<tr>
<td>Resolution</td>
<td>12 bit (or 4096 in division)</td>
</tr>
<tr>
<td>Maximum Sampling Rate</td>
<td>1.25 MS/s</td>
</tr>
<tr>
<td>Polarity</td>
<td>Unipolar or Bipolar</td>
</tr>
<tr>
<td>Board Range</td>
<td>Unipolar: 0 to 10V; Bipolar:-5V to 5V</td>
</tr>
<tr>
<td>Number of AO(^a) Channels</td>
<td>2</td>
</tr>
<tr>
<td>Trigger</td>
<td>AI(^b) channel &lt;0..15&gt;, external PFI&lt;0..9&gt;</td>
</tr>
</tbody>
</table>

\(^a\)AO = Analog output  
\(^b\)AI = Analog input  
\(^c\)PFI = Programmable Function Input

internally tied to the analog input ground so that the common ground noise can be rejected. The built-in multiplexer is an array of switching elements that route many input signals to one common output. Thus, only one cable needs to connect to the plug-in board in the computer. The analog-to-digital converter (ADC) controls the resolution, range and speed of the data acquisition device. Resolution is the number of bits that the ADC uses to convert the analog input signal. The PCI-MIO-16E-1 board has a resolution of 12 bit corresponding to 4096 quantization levels. Range refers to the maximum and minimum voltage levels that the ADC can quantize. The precision of an input signal is expressed as

\[
\text{precision} = \frac{\text{range}}{\text{gain} \times 2^N} 
\]  
(4.7)
where N is the resolution bit. The software selectable gain is controlled by PGIA. The effective range, $ER$, can be expressed as

$$ER = \frac{\text{range}}{\text{gain}}$$

(4.8)

Higher gain reduces the effective range, but enhances the precision. For instance, when the PCI-MIO-16E-1 board is set to bipolar, by adding a gain of 10, the precision becomes 244.14 $\mu$V. The gain-dependent conversion speed of the ADC is the time per conversion, or sample per second, and is usually provided by the manufacturer. ADC will cause interchannel delay when conducting multiple channel data acquisition. As illustrated in Figure 4.8, the switch of ADC from channel to channel results in asynchronization between channels within the sampling interval for each channel. Interchannel delay is also called settling time or time skew. The typical interchannel delay provided from the PCI-MIO-16E-1 manufacturer's specification is gain and accuracy dependent and ranges from 1 to 2$\mu$s. It is dependent on the conversion speed and can be removed by programming as long as the delay can be exactly determined.

4.4.3 The Control Program

Software

The control program is developed using the commercially available software LabView, a product of National Instrument Corporation. LabView is built upon a purely graphical, general purpose programming language, G, with extensive libraries of functions, an integral compiler and debugger, and an application builder for independent ap-
Figure 4.7: Configuration of the Data Acquisition Board

Figure 4.8: Illustration of Interchannel Delay
plications. The control program is designed as a virtual instrument which is able to perform data acquisition, pulse generation and data analyses.

Virtual instrument (VI) is a software-based, graphic instrument which mimics the control panel of a real instrument. Users can build a customized virtual panel which includes only the subset controls or displays of a real instrument. Instead of purchasing an oscilloscope, a spectrum analyzer or a waveform generator, a multi-function DAQ board and a computer with LabView can simulate all of these instruments. The control programs written for this system are virtual oscilloscope and virtual pulse generator. The virtual oscilloscope performs multichannel, triggered data acquisition and the virtual pulse generator controls the pulse that drives the vibration source.

**Virtual Oscilloscope**

The development of a virtual oscilloscope starts by designing a front panel that meet requirements. Required controls and displays are listed in Table 4.5 where the controls are equivalent to input parameters and the displays are the outputs of the program. The front panel shown in Figure 4.9 is the prototype of the virtual oscilloscope developed herein to perform 2 channel, triggered analog data acquisition. The acquired signals is processed and graphically displayed in terms of acquired waveforms, averaged waveform, and corresponding spectrum. The subordinate virtual instruments (subVIs) wired in the main VI are the programs that handles specific functions or operations that need to be performed. The FFT subVI converts an array of input time domain data into frequency domain using an FFT algorithm. The array average is performed by the AVG subVI in which the repetitively-read data are stored in the
buffer and averaged with the previous input arrays.

Table 4.5: Required Controls (Input) and Displays (Output) for the Oscilloscope VI

<table>
<thead>
<tr>
<th>Selectable</th>
<th>Constant</th>
<th>Graph</th>
<th>Indicator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Device</td>
<td>Buffer Size</td>
<td>Time Signal</td>
<td>Interchannel Delay</td>
</tr>
<tr>
<td>Input Channels</td>
<td>Repetition Done</td>
<td>Spectrum</td>
<td></td>
</tr>
<tr>
<td>Number of Scans</td>
<td>Trigger Type</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sampling Rate</td>
<td>Pretrigger Scans</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of Repetition</td>
<td>Trigger Channel</td>
<td></td>
<td></td>
</tr>
<tr>
<td>File Path</td>
<td>Trigger Source</td>
<td>Trigger Slope</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Trigger Level</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 4.9: The Front Panel of a 2-CH AI Oscilloscope
The programming of the virtual oscilloscope is composed of 4 parallel tasks which are repetitive data acquisition, data processing, displaying, and storing. All the tasks are programmed in terms of the block diagram shown in Figure 4.10. This block diagram also is the program's flow chart. Repetitive data acquisition is controlled by two nested WHILE loop. The inner WHILE loop handles the triggered data acquisition and instant display. A Logic control determines whether or not to accept the acquired waveform. The outer WHILE loop controls the buffered array that temporarily stores the acquired waveform. The termination of the outer WHILE loop is operated by a repetition counter. Once the repetitive data acquisition is complete, the outer WHILE loop stops executing, and the waveforms in the buffer are sent to the subordinate virtual instrument(subVI), AVGFFT, then averaged and transformed by the FFT subVI. The processed waveforms of each channel are displayed on the front panel in both time-domain and frequency domain. All the processed data are stored in spreadsheet format for later analysis and presentation.
Figure 4.10: The Block Diagram of the 2-Channel AI Virtual Oscilloscope
Virtual Pulse Generator

The front panel of the virtual pulse generator is shown in Figure 4.11. By running the pulse generator, the waveform displayed in the waveform graph will be converted into analog signals by the embedded ADC in the DAQ board. The converted signals are transmitted to the power amplifier and sent to the WR-F7 vibration shaker as the signal control source.

The control parameters for this virtual pulse generator include point update rate, number of points in one period, total number of updates, generation counts and voltage limits. Figure 4.12 illustrates the definition of the input parameters for the virtual pulse generator. Point update rate defines the time interval between two generated vibration spikes. The number of points in one period gives the composition of waveform. Total number of updates and generation counts tells the length of the pulse or the number of wave cycles included in the pulse. The example in Figure 4.12 shows a 4 cycle square pulse wave where each cycle is composed of 2 updates. The update rate is the inverse of the time interval between two updates. The magnitude of the output pulse can be controlled by adjusting the voltage ranges.
Figure 4.11: The Front Panel of the Pulse Generator
Figure 4.13 shows the block diagram of the virtual pulse generator. The major subVIs in the program are a data generator, an AO waveform generator and an AO writer. These subVIs are modified from the VIs appended to the LabView package. The data generation subVI is used to create a square wave of the specified length. The analog output waveform generator subVI is capable of creating a timed, buffered waveform at a specified update rate. The AO writer subVI writes the data into the buffer for the buffered analog output operation (National-Instrument, 1998).

Figure 4.12: Illustration of the Input Parameters for the Pulse Generator:
The example shows that each period is composed of 2 points and 4 periods in total.
Figure 4.13: The Block Diagram of the Pulse Generator
4.5 System Integration

4.5.1 Integration of Subsystem

Figure 4.14 schematically shows the integrated test system to perform waveform generation and vibration measurement. The system is composed of a portable PC control system, vibration generation system and measurement assembly. The portable PC control system is built on a portable computer with plug-in DAQ boards and I/O control programs developed from LabView software. The vibration generation system consists of 3 components: a piezoelectric shaker, a power amplifier and a matching network. Waveforms generated from the PC control system are amplified and sent to the shaker as a control signal. Excited responses are measured by accelerometers powered by the signal conditioner. Measured signals are sent back to the PC control system through multi-channel triggered data acquisition process. The PC control system conducts post-run signal processing and data analysis. Accepted data are stored in the hard drive for advanced analysis and presentation. It is noted that the I/O connector used in the test system is a NI-BNC-2080 connection block which contains 16 analog single end AI channels and 2 AO output channels.

4.5.2 Test Operation

The PC control system is capable of parallel running the two virtual instruments, a pulse generator and an oscilloscope, in a manner as if using two real instruments to perform waveform generation and control data acquisition. The general procedures for operating the PC controlled system are summarized as follows:

1. The integrated system must be made ready to start the test by mounting ac-
celerometers and a vibration shaker to their planned locations, appropriately connecting each subsystem, and turning on the power of each equipment.

2. When an impact hammer is used as the vibration source, one opens the virtual oscilloscope. Appropriate control parameters are set and the AI program is started. When a modal shaker is used as the vibration source, one opens both the virtual oscilloscope and pulse generator. Appropriate control parameters are selected and the oscilloscope and pulse generator are sequentially run.

3. The oscilloscope will not start data acquisition until the received signals from the output of the pulse generator fulfill the requirements of the assigned trigger event. The general trigger event for a particular application is composed of the following parameters: device(DAQ Board), analog, pretrigger scans, rising edge, and trigger level. For example, the start of data acquisition can be triggered by the event that rising analog input signal reaches the level of 1 volt. The pretrigger scans representing the state of the signals before the trigger event will be included in the acquired data.

4. The acquired waveform is displayed on the virtual oscilloscope instantly. The waveforms are checked and a decision is made whether or not to accept the acquisition.

5. After the repetitive data acquisition is complete, post-run data processing is started and average waveform and corresponding spectrum are displayed.
Figure 4.14: Schematic of Test System
Chapter 5

Experimental Evidence of Guided Wave Theory in Conventional Impact Tests

5.1 Introduction

Guided wave theory described in Chapter 2 implies the results of the impulse response test can be evaluated by the dispersion relations of the first longitudinal branch. However, the numerically-developed theory applicable to the results of the impulse response test has not been experimentally verified. As part of this research, conventional impulse response tests have been conducted on a number of prototype cylindrical concrete piles to provide evidence of frequency-dependent responses predicted by the guided wave approach. These prototype piles were constructed in the laboratory and installed at the NGES site at Northwestern University.

The parameters that affect the guided wave results including the shear modulus and density ratios between the pile and the surrounding soil and the Poisson’s ratio
of the pile and the soil, were experimentally determined for input to the theoretical non-dimensional dispersion relations. The measured results and the non-dimensional numerical solutions are normalized by the bulk shear wave velocity of the pile so that comparisons can be made directly. Methods for determining the bulk shear wave velocity are proposed and evaluated based on the $L/D$ ratio and the useful resonant frequencies. The dispersion of the measured phase velocities were evaluated on piles subjected to both traction-free and embedded boundary conditions. The geometrical attenuation as a result of the radially-lost energy was evaluated by comparing the theoretical attenuation coefficients with the experimentally-measured attenuation from the results of tests on both traction-free and embedded piles.

5.2 Prototype Piles

The effects of pile diameter on wave dispersion are evaluated by prototype piles with diameters varying from 152 to 457 mm. As a result of the interactions between the propagating waves and the boundaries of the wave-guide, the waves are dispersive in nature. The dispersion is a function of the wavelength and the diameter of a cylindrical wave guide. The accuracy for measuring the propagating velocity is dependent on the resolution time signals. If the time resolution of the acquired data is limited, the waveguide with longer length is preferred such that the measurement error can be reduced. For practical consideration, the length to diameter ratio ($L/D$) of the pile has to exceed a minimum value, and thus the prototype piles cast in the laboratory are with the minimum $L/D$ ratio is 4 for the 457-mm diameter pile and $L/D$ ratios from 5.96 to 8.71 for the other prototype piles.
The prototype piles were cast in laboratory using the commercially available fiber tube. This form is made from many layers of high-quality fiber, spirally wound and laminated with special adhesives. The manufacturer suggests that the tubes will not buckle, swell, or lose shape for concrete pressure up to 3000 lb per sq ft. Concrete specimen were made and cured following procedures specified in ASTM C192-88. The density can be determined from direct measurement.

5.2.1 Group A Piles

Two sets of prototype piles are constructed. The first set of prototype piles, summarized in Table 5.1, is called “group A”. The schematics of these four prototype piles are given in Figure 5.1. The A06-091 and A08-122 piles were constructed to be “intact” and with constant cross section. The “06” after A refers to the pile diameter in inches and the “091” refers to this pile length in centimeters. The A06-101 pile was built with a 25 mm deep notch circumferentially at L=0.81 m from the top of the pile. This circumferential notch is analogue to the defect “necking” which may occur in the construction of drilled shaft. The A06-131 is a composite structure composed of two section of intact cylinders laminated together by epoxy. The length of the top section is 100 cm while that of the bottom section is 31 cm. This discontinuity is used to simulate the “soil intrusion” that sometimes occurs due to inappropriate construction procedure. The group A piles are small, easy to move, and are used to evaluate wave propagation in traction-free boundary condition.
The traction-free boundary prevents the wave energy from “leaking” out the side of the piles, and thus the dispersion behavior of the guided waves can be evaluated without the consideration of geometrical attenuation. Wave propagation in intact piles is evaluated by tests on piles A08-122 and A06-091 while the effects of a notch and a discontinuity are evaluated by the prototype piles A06-101 and A06-131.

Table 5.1: Table of Prototype Piles- Group A

<table>
<thead>
<tr>
<th>Pile ID no.</th>
<th>Diameter D (m)</th>
<th>Length L (m)</th>
<th>$\frac{L}{D}$</th>
<th>Density $\rho$ (kg/m$^3$)</th>
<th>Design Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td>A08-122</td>
<td>0.203</td>
<td>1.22</td>
<td>6.01</td>
<td>2350</td>
<td>intact</td>
</tr>
<tr>
<td>A06-091</td>
<td>0.152</td>
<td>0.91</td>
<td>5.99</td>
<td>2340</td>
<td>intact</td>
</tr>
<tr>
<td>A06-101</td>
<td>0.152</td>
<td>1.01</td>
<td>6.64</td>
<td>2408</td>
<td>25 mm deep notch</td>
</tr>
<tr>
<td>A06-131</td>
<td>0.152</td>
<td>1.31</td>
<td>8.61</td>
<td>2335</td>
<td>discontinuity at L=1.00m</td>
</tr>
</tbody>
</table>

*Measured directly on the prototype piles

5.2.2 Group B Piles

Another set of prototype piles is called “group B” and is summarized in Table 5.2. The same notation as for the A series piles is used to identify the B piles. These piles are larger and are used to evaluate the embedded boundary condition. The group B piles are cylinders with diameters of 254, 305 and 457 mm and lengths of 1.73 to 2.40 m. The effect of surrounding soils on the wave propagation is to cause geometrical attenuation of the guided waves. Group B piles were constructed in the laboratory...
Figure 5.1: Schematic of Group A Prototype Piles (not to scale)

and installed later in the National Geotechnical Experimental Site (NGES) at NU. Based on numerical evaluation, the larger diameters of the group B piles result in wave dispersion occurring at lower frequencies than would occur in the A series piles. The geometrical attenuation should be mobilized more easily in group B pile because they are longer than group A.

5.2.3 Material Properties Based on Concrete Cylinder Specimen

Concrete material properties for the prototype piles were estimated based on tests conducted on specimens made from the concrete mixes that comprise the prototype piles. Table 5.3 lists the concrete mixes for group A and group B piles. The dynamic
Table 5.2: Table of Prototype Piles- Group B

<table>
<thead>
<tr>
<th>Pile ID no.</th>
<th>Diameter</th>
<th>Length</th>
<th>( \frac{L}{D} )</th>
<th>Density(^a)</th>
<th>Design Feature</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>D (m)</td>
<td>L (m)</td>
<td>( \rho ) (kg/m(^3))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B10-222</td>
<td>0.254</td>
<td>2.22</td>
<td>8.74</td>
<td>2375</td>
<td>intact</td>
</tr>
<tr>
<td>B12-182</td>
<td>0.305</td>
<td>1.82</td>
<td>5.96</td>
<td>2320</td>
<td>intact</td>
</tr>
<tr>
<td>B12-240</td>
<td>0.305</td>
<td>2.40</td>
<td>7.87</td>
<td>2365</td>
<td>intact</td>
</tr>
<tr>
<td>B18-173</td>
<td>0.457</td>
<td>1.73</td>
<td>3.79</td>
<td>2401</td>
<td>intact</td>
</tr>
</tbody>
</table>

\(^a\)Measured from ASTM C198-22 specimen

elastic constants can be estimated by ASTM C215. However, the assumption that the Poisson’s ratio is equal to \( \frac{1}{6} \) is implicit when using this method. Subramaniam et al. (2001) proposed a method for determining elastic constants of concrete based on the three-dimensional Rayleigh-Ritz model. The Young’s Modulus, \( E \), and Poisson’s ratio, \( \nu \), can be calculated by the measured fundamental \( f_0 \) and first longitudinal harmonic \( f_1 \) resonant frequencies from the FFT transformed data of impact echo test on an intermediate-length \( 1.5 < \frac{L}{D} < 2.5 \) traction-free cylinder from:

\[
\nu = A_1 \left( \frac{f_1}{f_0} \right)^2 + B_1 \frac{f_1}{f_0} + C_1 \quad (5.1)
\]

where \( A_1, B_1 \) and \( C_1 \) are functions of length to diameter ratio, \( \frac{L}{D} \). This value of Poisson’s ratio is then used to calculate the Young’s modulus, \( E \), from:

\[
E = 2(1 + \nu) \rho \left( \frac{\pi f_0 d}{\alpha} \right)^2 \quad (5.2)
\]
Table 5.3: The Concrete Mixes for Group A and Group B Piles

<table>
<thead>
<tr>
<th>Piles</th>
<th>Mixing Ratio (by weight)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Cement</td>
<td>Water</td>
<td>FA&lt;sup&gt;a&lt;/sup&gt;</td>
<td>CA&lt;sup&gt;b&lt;/sup&gt;</td>
</tr>
<tr>
<td>A08-122</td>
<td>1</td>
<td>0.5</td>
<td>0.9</td>
<td>1.8</td>
</tr>
<tr>
<td>All other Group A piles</td>
<td>1</td>
<td>0.5</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>All group B piles</td>
<td>1</td>
<td>0.5</td>
<td>2.0</td>
<td>2.3</td>
</tr>
</tbody>
</table>

<sup>a</sup>fine aggregate  
<sup>b</sup>coarse aggregate

where

\[
\alpha = A_2 \nu^2 + B_2 \nu + C_2
\]  

(5.3)

and \(A_2, B_2\) and \(C_2\) are also functions of \(\frac{L}{D}\). Subramaniam et al. (2001) presented the following values for these coefficients for cylindrical concrete specimens with a \(\frac{L}{D}\) ratio between 1.8 to 2.2:

\[
\begin{align*}
A_1 &= -8.6457 \left( \frac{L}{D} \right)^2 + 24.4431 \left( \frac{L}{D} \right) - 12.4778 \\
B_1 &= 34.5986 \left( \frac{L}{D} \right)^2 - 101.7207 \left( \frac{L}{D} \right) + 56.1720 \\
C_1 &= -34.6807 \left( \frac{L}{D} \right)^2 + 105.9790 \left( \frac{L}{D} \right) - 62.7310 \\
A_2 &= -0.2792 \left( \frac{L}{D} \right)^2 + 0.4585 \left( \frac{L}{D} \right) - 2.1093 \\
B_2 &= 0.0846 \left( \frac{L}{D} \right)^2 - 0.5868 \left( \frac{L}{D} \right) + 1.3791 \\
C_2 &= 0.2850 \left( \frac{L}{D} \right)^2 - 1.7026 \left( \frac{L}{D} \right) + 3.3769
\end{align*}
\]  

(5.4)
The shear modulus of the cylindrical specimen can then be calculated by the Poisson's ratio and Young's modulus from:

\[ G = \frac{E}{2(1 + \nu)} \]  

(5.5)

The elastic constants based on the Popovics and Wang (1999) method for each cylindrical specimen obtained from the group B piles are listed in Tables 5.4 and 5.5. All of the group B piles were cast with concrete with the same mixing design. The B12-182 and B18-173 prototype piles were poured at the same time and the C10-222 and C12-240 are poured together at another time.

<table>
<thead>
<tr>
<th>Specimen</th>
<th>( f_0 ) (Hz)</th>
<th>( f_1 ) (Hz)</th>
<th>( \frac{f_2}{f_1} )</th>
<th>( \frac{L}{D} )</th>
<th>( \nu )</th>
<th>E (GPa)</th>
<th>G (Gpa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>13087</td>
<td>23444</td>
<td>1.7914</td>
<td>1.94</td>
<td>0.44</td>
<td>75.2</td>
<td>26.1</td>
</tr>
<tr>
<td>S2</td>
<td>13076</td>
<td>26087</td>
<td>2.00</td>
<td>1.94</td>
<td>0.06</td>
<td>36.5</td>
<td>17.2</td>
</tr>
<tr>
<td>S3</td>
<td>13064</td>
<td>24060</td>
<td>1.84</td>
<td>1.95</td>
<td>0.33</td>
<td>53.5</td>
<td>20.1</td>
</tr>
</tbody>
</table>

Table 5.4 lists the mean, standard deviation and variance of the Poisson's ratio and shear modulus measured from the specimens. Although the means of these elastic constants are reasonable for concrete, the standard deviation and variance imply significant uncertainty may arise when applying the data based on the test cylinders directly to the prototype piles. These values of Poisson's ratio are very sensitive to a change in the measured resonant frequency.
Table 5.5: B10-222 and B12-240: Poisson’s Ratio and Shear Modulus from Batch 2 Specimens Using Popovics and Wang Approach (1999)

<table>
<thead>
<tr>
<th>Specimen</th>
<th>$f_0$ (Hz)</th>
<th>$f_1$ (Hz)</th>
<th>$\frac{f_1}{f_0}$</th>
<th>$\frac{L}{D}$</th>
<th>$\nu$</th>
<th>E (GPa)</th>
<th>G (Gpa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S4</td>
<td>12771</td>
<td>25609</td>
<td>2.0052</td>
<td>2.00</td>
<td>0.05</td>
<td>36.6</td>
<td>17.5</td>
</tr>
<tr>
<td>S5</td>
<td>7126</td>
<td>12771</td>
<td>1.7922</td>
<td>1.99</td>
<td>0.46</td>
<td>26</td>
<td>8.9</td>
</tr>
<tr>
<td>S6</td>
<td>11899</td>
<td>22342</td>
<td>1.8776</td>
<td>2.00</td>
<td>0.28</td>
<td>42.2</td>
<td>16.5</td>
</tr>
</tbody>
</table>

Table 5.6: Uncertainty of the Measurements

<table>
<thead>
<tr>
<th>Specimen</th>
<th>Poisson’s ratio</th>
<th>Shear Modulus (GPa)</th>
<th>$a\bar{\nu}$</th>
<th>$b\sigma$</th>
<th>$c\sigma^2$</th>
<th>$G\bar{\sigma}$</th>
<th>$\sigma$</th>
<th>$\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Batch 1</td>
<td>0.28</td>
<td>0.20</td>
<td>0.04</td>
<td>21.13</td>
<td>4.54</td>
<td>20.60</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Batch 2</td>
<td>0.26</td>
<td>0.21</td>
<td>0.04</td>
<td>14.31</td>
<td>4.69</td>
<td>21.96</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*a*mean  
*b*standard deviation  
*c*variance
5.3 Sand Fill at the NGES Site at NU

The piles were embedded in the sand fill at the NU NGES. The Poisson's ratio, $\nu_s$, of the sand fill is assumed to be 0.30, although it can be calculated by the shear wave velocity $c_T$ and propagation velocity $c_p$ using:

$$\nu_s = \frac{1 - \frac{a^2}{2}}{1 - a^2}, \quad a = \frac{c_p}{c_T}$$  \hspace{2cm} (5.6)

The dynamic methods to measure $c_T$ and $c_p$ include crosshole seismic surveys, seismic downhole surveys, seismic cone penetration(CPT) tests, and spectral analysis of surface waves(SASW) tests. The Poisson's ratio determined by this expression, however, is rather unreliable: small error in the values of $c_p$ or $c_T$ will lead to substantial errors in $\nu_s$. The numerical evaluation shown in Figure 5.2 implies that the wave propagation velocity in the pile is essentially independent of $\nu_s$. At low value of $\Omega$, $\nu_s$ has insignificant effect on the geometrical attenuation of the L(0,1) mode. At values of $\Omega$ greater than 2, $\nu_s$ affects geometrical attenuation in that the smaller the $\nu_s$, the lower the damping. Based on these considerations, using a reasonable predicted $\nu_s$ to develop dispersion curves should lead to reasonable results, with the exception of the geometric damping at values of $\Omega$ greater than 2. The group B prototype piles are embedded in a dry, fine-grained sand. The Poisson's ratio, $\nu_s$, is assumed to be 0.30 throughout this study.

The dynamic shear modulus, $G_{max}$, can be computed from the shear wave velocity,
Figure 5.2: The effect of the Poisson's ratio of soil, $\nu_s$ on the dispersion relations for embedded cylindrical waveguide: The real part of the dispersion relations is practically unaffected by the Poisson's ratio of the surrounding soil while the imaginary part of the dispersion relations is a function of the Poisson's ratio of the surrounding soils.
\[ G_{\text{max}} = \rho_s \cdot c_{T_s}^2 \]  

where \( \rho_s \) is the total mass density of the soil. Results of cross-hole seismic tests (Gassman, 1997) indicated the corresponding shear wave velocity varies from 110 to 115 m/s and the density ranges from 1500 to 1550 kg/m\(^3\). The corresponding shear modulii ranges between 18.2 and 20.5 MPa based on Equation 5.7. The shear modulus ratio between group B prototype piles and the surrounding soils is computed to be between 786 to 698 based on a mean shear modulus of 14.3 GPa for the group B prototype piles. The parameters used to develop dispersion curves are summarized in Table 5.7.

<table>
<thead>
<tr>
<th>Prototype Pile</th>
<th>( \nu_p )</th>
<th>( \nu_s )</th>
<th>( \frac{\mu_s}{\rho_s} )</th>
<th>( \frac{\rho_s}{\rho_s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.28</td>
<td>--</td>
<td>( \infty )</td>
<td>( ^a2400 )</td>
</tr>
<tr>
<td>B</td>
<td>0.26</td>
<td>0.30</td>
<td>740(^b)</td>
<td>1.5(^c)</td>
</tr>
</tbody>
</table>

\(^a\)traction-free boundary condition  
\(^b\)The median of the estimated \( \frac{\mu_s}{\rho_s} \)  
\(^c\)The median of the estimated \( \frac{\rho_s}{\rho_s} \)
5.4 Non-dimensional Dispersion Relations Applicable to Conventional Impact Tests

Figure 5.3 illustrates the operation of the numerical model for developing the dispersion curves for a pile. The numerical model was programmed with Maple, a software package of Waterloo Maple Inc. The input parameters for this model include $\nu_p$, $\nu_s$, $\frac{\rho_p}{\rho_s}$, $\frac{\mu_p}{\mu_s}$. The output of the model is the dispersion relation in terms of non-dimensional frequency, $\Omega$ versus non-dimensional wave number, $\xi a$. The non-dimensional phase velocity, $C_p$, non-dimensional group velocity, $C_g$ and non-dimensional geometrical attenuation, $\Xi$, dispersion curves can thereafter be generated by the same program in the following manner:

$$C_p = \frac{\Omega}{\xi a} \quad (5.8)$$

$$C_g = \frac{d\Omega}{d(\xi a)} \quad (5.9)$$

$$\Xi = \xi a \quad (\text{neper}) \quad (5.10)$$

Figure 5.4 shows the non-dimensional phase velocity, group velocity and geometrical attenuation coefficient dispersion curves for the L(0,1) branch developed from the pile and soil parameters listed in the Table 5.7. The shear modulus ratio and density ratio between the pile and soil are 740 and 1.5, respectively. The Poisson’s ratio of surrounding soil is approximated to be 0.3. To reduce the uncertainty caused by minor changes of the properties of the concrete specimen in determining the value of the Poisson’s ratio, additional dispersion curves were developed based on $\nu_p$ of 0.14,
Figure 5.3: Illustration of the numerical dispersion curve computation
0.18, 0.20, 0.25 and 0.28.

The conversion between the non-dimensional and dimensional solutions of $f$, $c_p$, and $c_g$ is made by substituting the bulk shear wave velocity of the pile, $c_T$, into the following equations:

\[
f = \frac{c_T \cdot \Omega}{2\pi a} \quad (5.11)
\]
\[
c_p = C_p \cdot c_T \quad (5.12)
\]
\[
c_g = C_g \cdot c_T \quad (5.13)
\]

where the $c_p$ and $c_g$ represent the phase velocity and group velocity, respectively, with the same units as $c_T$. The dimensional solution of $\xi_i$ can be derived from dividing the non-dimensional geometrical attenuation coefficient, $\Xi$, by pile radius, $a$.

\[
\xi_i = \frac{\Xi}{a} \quad (5.15)
\]

The bulk shear wave velocity can be computed from the Poisson’s ratio and the bar wave velocity, $c_B$, of the pile using:

\[
c_T = \frac{c_B}{\sqrt{2 \cdot (1 + \nu)}} \quad (5.16)
\]

Generally, for piles with a large L/D ratio, the bar wave velocity is approximately equal to the propagation velocity measured from the time-domain waveform. The Poisson’s ratio is determined from the method introduced in Section 5.2.3 using the
Figure 5.4: The L(0,1) branch of group velocity, phase velocity, and attenuation dispersion curves developed for soil-pile system having the parameters: $\nu_s=0.3$, $\frac{\mu_p}{\mu_s}=740$, $\frac{\rho_p}{\rho_s}=1.5$
concrete specimens made while constructing the piles.

5.5 Experimental Procedures

The impact echo tests were conducted on both traction-free and embedded piles. The traction-free boundary condition was evaluated by the group A piles and the selected group B piles. The embedded boundary condition was evaluated by group B piles. As introduced in Section 4.5, the test system for the impulse response test is composed of the following components:

1. The portable PC and the PC control program (Sections 4.4.2 and 4.4.3).

2. Two PCB-352A78 Accelerometers (Section 4.2.3).

3. The PCB-482-A20 signal conditioner and the NI-BNC-2080 connection block (Section 4.2.3).

4. The PCB-086C04 Modal hammer (Section 4.2.1).

Figure 5.5 illustrates the test setup. An impulse generated by a hammer impact is introduced into the prototype pile from its top surface. Shear wave, surface wave, and the longitudinal waves are all excited by the impulse hammer. Nevertheless, the uniaxial accelerometers will primarily pick up the vibrations of the longitudinal modes of waves if the mounting is ideally oriented. The responding vibrations are measured at two or three locations across the top surface of the pile. The signals acquired at each location were averaged three times before data were analyzed. The PC control system is configured as a single ended (SE), 3 or 4-channel analog input (AI) synchronized data acquisition system. The trigger event is controlled by the
hammer impact so that the start time of data acquisition can be identified from the starting point of the rising edge in the plot of impact force versus time response.

The sampling rate is set to 100 kHz at which the spatial resolution is 4 cm when the propagation velocity is 4000 m/s. The number of samples acquired is 9192 so that the length of data is 91.92 ms and the frequency resolution is 10.88 Hz.

![Diagram of setup](image)

* : $R = \pi a$, where $a$ is the radius and $r$ is the distance to the center.
** : Trigger channel

Figure 5.5: The setup of the impact-echo test

5.6 Experiment Results: Group A Piles

Figures 5.6 to 5.9 show the input force response and force spectrum introduced by the hammer-impact, the measured acceleration responses at assigned locations, and the accelerance for intact piles A06-091 and A08-122 and defective piles A06-101 and
A06-131. The force responses of the hammer impacts are shown in part (a) of the Figures 5.6 to 5.9. The duration of the hammer impact controls the energy introduced into the structure (Carino and Sansalone, 1986). The maximum frequency, $f_{\text{max}}$, that the propagation waves can be carried through the structure is estimated to be:

$$f_{\text{max}} = \frac{1}{\Delta t_h}$$ \hspace{1cm} (5.17)

where $\Delta t_h$ is the duration of the impact. Table 5.8 lists $\Delta t_h$ and the maximum useful frequency for each of the tests. Although all the tests are conducted by the same hammer, PCB 086C04, variations in impact durations are still observed, as a result of variations in the impact velocity, angle of the impact, and the spring constant of the impacted surface. The maximum suggested useful frequencies varied from 2040 to 3333 Hz.

<table>
<thead>
<tr>
<th>Pile</th>
<th>$\Delta t_h$ (ms)</th>
<th>$f_{\text{max}}$ (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-122</td>
<td>0.37</td>
<td>2700</td>
</tr>
<tr>
<td>A06-091</td>
<td>0.32</td>
<td>3125</td>
</tr>
<tr>
<td>A06-101</td>
<td>0.30</td>
<td>3333</td>
</tr>
<tr>
<td>A06-131</td>
<td>0.49</td>
<td>2040</td>
</tr>
</tbody>
</table>

Table 5.8: The Maximum Introduced Frequency for Group A Prototype Piles

Parts (c) of Figures 5.6 to 5.9 show the accelerations of the surface vibrations measured at $R=0.5$ and $R=0.75$. The travel time, $(\Delta t)_{avg}$, for the waves propagating down and back the prototype pile, is determined from the intervals between the
Figure 5.6: Pile No. A08-122: Results of Impulse Response Test. (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at R=0.5 and 0.75, (d) The frequency responses of the pile measured at R=0.5 and 0.75.
Figure 5.7: Pile No. A06-091: Results of Impulse Response Test. (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at R=0.5 and 0.75, (d) The frequency responses of the pile measured at R=0.5 and 0.75.
Figure 5.8: Pile No. A06-101 (with Notch): Results of Impulse Response Test. (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at R=0.5 and 0.75, (d) The frequency responses of the pile measured at R=0.5 and 0.75.
Figure 5.9: Pile No. A06-131 (with a Discontinuity): Results of Impulse Response Test. (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at $R=0.5$ and 0.75, (d) The frequency responses of the pile measured at $R=0.5$ and 0.75.
waveform peaks. The first peak in the waveform is neglected in the interpretation because it results from the arrival of the Rayleigh wave or the wave front of the introduced waveform instead of a reflection of the longitudinal wave. The propagation velocities are then calculated by:

\[ c = \frac{2L}{\Delta t} \]  

(5.18)

The calculated propagation velocities are listed in column (4) of Table 5.9.

The frequency response of each prototype pile is expressed in terms of the accelerance and is shown in part (d) of the figures 5.6 to 5.9. The accelerance is defined as the measured acceleration divided by the input force over the frequency range being excited.

\[ A(\omega) = \frac{a(\omega)}{F(\omega)} \]  

(5.19)

The average propagation group velocity for waves within the useful frequency range can be computed from the average resonant interval using

\[ c = 2(\Delta f)_{avg} L \]  

(5.20)

The propagation group velocities computed by Equation 5.20 are listed in column (6) of Table 5.9. The phase velocity for wave propagating at its resonant frequency is estimated by

\[ c = f \cdot \lambda \]  

(5.21)

where \( f \) is the measured resonant frequency and \( \lambda \) is the corresponding wave length
determined by its harmonic number, $n$.

$$
\lambda = \frac{2L}{n}
$$

(5.22)

The calculated phase velocity at each of the resonant frequencies is listed in column (11). The averaged phase velocity for each prototype piles is given in column (12). Result of the comparison between the group velocity in column (4) and the phase velocity in column (12) suggests that the phase velocity equals to the group velocity for the waves being excited by hammer impact. This observation is consistent with the results evaluated from the guided wave approach for \( L(0,1) \) branch at \( \Omega < 1.41 \) which is equivalent to the frequencies 5820Hz for pile A08-122, 6897Hz for pile A06-091, 6820Hz for pile A06-101, and 7248 for pile A06-131 based on the shear wave velocities as will be determined later in this chapter. Note that the "useful" frequencies, as noted by the \( f \) corresponding to \( n=4 \) in column 8 of Table 5.9 are larger than those given in Table 5.8. This suggests the \( f_{\text{max}} \) computed by Equation 5.17 is a conservative estimate.
Table 5.9: The Impact Echo Test Results of Group A Prototype Piles

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>Length, L (m)</th>
<th>Time Domain Response</th>
<th>Frequency Domain Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Δt_{avg} (ms)</td>
<td>c_g (m/s)</td>
</tr>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
</tr>
<tr>
<td>A06-122</td>
<td>1.22</td>
<td>0.61</td>
<td>4000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A06-091</td>
<td>0.91</td>
<td>0.51</td>
<td>3569</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A06-101</td>
<td>1.01</td>
<td>0.56</td>
<td>3607</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A06-131</td>
<td>1.31</td>
<td>0.71</td>
<td>3690</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.6.1 Intact Group A Piles

Figure 5.10 shows the measured phase velocities at resonant frequencies superimposed on the dispersion curves developed for the prototype piles A06-091 and A08-122 converted from the shear wave velocities computed by a number of values of Poisson’s ratio. As can be seen, the trend of the measured phase velocities approximate the dispersion curves developed from \( \nu = 0.15 \), and not the Poisson’s ratio of 0.28 determined from the test specimens obtained while making the prototype piles. This discrepancy may be a result of the following:

1. The Poisson’s ratio is very sensitive to minor changes in the material properties; the quantity measured from the test specimens may not be representative of the Poisson’s ratio of the prototype piles.

2. The theoretical guided wave approach presented in Chapter 2 was developed based on the assumption that a harmonic wave excited by an uniformly distributed harmonic vibration source propagates along an infinitely long cylindrical wave-guide. This theoretical background is essentially different from the realistic experimental condition that a stress wave generated by a point impact source propagates along a finite length cylindrical wave-guide.

Because the Poisson’s ratio determined from the test specimens may not be reliable, it is necessary to identify this quantity directly from the prototype piles being tested. The “best match” approach based on the assumption that the bar wave velocity, \( c_B \), is equal to the propagation velocity measured in the time domain is proposed for estimating the Poisson’s ratio and shear wave velocity of the piles being tested herein. This approach is performed by the following procedure:
1. Compute the bulk shear wave velocity $c_T$ by Equation 5.16 from a number of Poisson's ratios within a reasonable range for the concrete material. The bar wave velocity, $c_B$ is assumed to be equal to the propagation velocity measured in the time domain as listed in column (6), Table 5.9.

2. Convert the non-dimensional phase velocity dispersion curves into the dimensional solutions in terms of velocity(m/s) versus frequency(Hz) using the computed shear wave velocities.

3. Superimpose the the measured phase velocities on the same plot of the converted dispersion curves. The dispersion curve that has the least variation to the measured data is identified as the characteristic dispersion curve for the pile being tested. The Poisson's ratio and the shear wave velocity of the pile can thus be estimated.

A number of the possible shear wave velocities for A08-122 and A06-091 computed based on the assumed Poisson's ratios are listed in Table 5.10. Figure 5.10 shows the converted phase velocity dispersion curves and the superimposed measurement data. As can be seen, the dispersion curves with $\nu$ of 0.15 for A08-122 and $\nu$ of 0.16 for A06-091 exhibit the least amount of variation between the numerical and experimental data. Thus, the shear wave velocity are 2635m/s and 2342m/s for the piles A08-122 and A06-091, respectively.

Although the "best match" approach can be performed based on a finite number of data points, it is necessary to extend the testing frequency range when one is attempting to verify the guided wave theory experimentally. At higher frequencies,
Figure 5.10: Estimate the shear wave velocity and Poisson’s ratio for piles A08-122 and A06-091 by the best match approach based on the results of impulse response test and the numerically-derived L(0,1) branch phase velocity dispersion curves.
Table 5.10: The Shear Wave Velocity Approximation for A08-122 and A06-091

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>$c_B$ (m/s)</th>
<th>$\nu$</th>
<th>$c_T$ (m/s)</th>
<th>Pile No.</th>
<th>$c_B$ (m/s)</th>
<th>$\nu$</th>
<th>$c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>3569</td>
<td>0.14</td>
<td>2364</td>
<td></td>
<td></td>
<td>0.16</td>
<td>2626</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.20</td>
<td>2304</td>
<td></td>
<td>4000</td>
<td>0.20</td>
<td>2604</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.23</td>
<td>2276</td>
<td></td>
<td>0.23</td>
<td>0.23</td>
<td>2582</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.28</td>
<td>2231</td>
<td></td>
<td>0.28</td>
<td>2500</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.16a</td>
<td>2342</td>
<td></td>
<td>0.15b</td>
<td>0.15b</td>
<td>2635</td>
</tr>
</tbody>
</table>

$^a$"Best fit" result
$^b$"Best fit" result

presumably the phase velocity will show a more pronounced variation with frequency.

5.6.2 Defective Group A Piles

The wave propagation along a prototype pile with a known defect is evaluated by conducting tests on piles A06-101 and A06-131. As shown in Figure 5.1, the A06-101 pile has an 25 mm deep notch around its perimeter at 0.81m from the top of the pile, and the A06-131 pile is a composite structure of two separate cylinders with lengths of 100cm and 31cm laminated together. The notch for the A06-101 pile is considered as the "necking", and the laminated part in A06-131 is considered similar to a discontinuity caused by soil intrusion. These defects occasionally occur as drilled shafts are constructed.

Figure 5.8 and 5.9 show the input force response and force spectrum introduced
by the hammer-impact, the measured acceleration responses at assigned locations, and the accelerance for piles A06-101 and A06-131. Table 5.11 lists the propagation velocities measured from the time domain waveforms based on the primary reflection peaks. Assuming \( c_B \) equal to \( c_p \), the bulk shear wave velocity \( c_T \) can be computed by Equation 5.16. The non-dimensional phase velocity dispersion curves can thereafter be converted into the velocity(m/s) versus frequency(Hz) by applying the computed shear wave velocities.

Table 5.11: The Shear Wave Velocity Approximation for A06-101 and A06-131

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>( c_p = c_B )(m/s)</th>
<th>Assumed ( \nu )</th>
<th>Computed ( c_T ) (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-101</td>
<td>3607</td>
<td>0.16</td>
<td>2368</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.28</td>
<td>2306</td>
</tr>
<tr>
<td>A06-131</td>
<td>3690</td>
<td>0.16</td>
<td>2423</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.28</td>
<td>2404</td>
</tr>
</tbody>
</table>

Extra reflection peaks and resonances that were not present in the results of intact piles A06-091 and A08-122 are now found in the results of the defective piles A06-101 and A06-131, as shown in Figures (c) and (d) of 5.8(c),(d) and 5.9. In these figures, the primary reflections are denoted as \( P_n \) and the secondary reflections are denoted as \( S_n \) where \( n \) represents the order of reflections. The averaged time interval between the primary reflections is represented by \( (\Delta t_P)_{avg} \), and the averaged time interval between secondary reflections is represented by \( (\Delta t_S)_{avg} \) while the time interval between \( P_n \) and \( S_n \) is denoted by \( \Delta t_{SP_n} \).
For pile A06-101, the propagation velocity of the primary reflection waves as shown in Figure 5.8(c) was computed to be 3607m/s based on the averaged time interval of 0.56ms and the assumption that the primary reflection source is the bottom of the pile. The measurement location at R=0.5 is essentially above the inner part of the 25mm deep notch while at R=0.75 is above the circumferentially discontinuous part of the notch. The accelerations of the surface vibration as a result of the first two primary reflections, $P_1$ and $P_2$, measured at R=0.5 and 0.75 showing the same magnitude which implies these primary reflections were not affected by the location and size of the notch. The secondary reflection peaks, $S_i$, suggest the existence of an anomaly within the pile. It can be inferred that the geometry of the anomaly is not uniform across the cross section because the more complicated signals were observed from the signals measured above the outside notch, R=0.75. However, the location of the defect cannot be identified based on the measured time interval $\Delta t_{sp}$ of 0.2ms and the propagation velocity 3607m/s. The propagation distance between the primary and secondary reflection sources were computed to be 72cm which is not close to the expected 40cm, twice of the distance between the bottom of the pile and the notch. This discrepancy is possibly the result that the wavelength generated by a hammer-impact is much larger than the size of the notch. To increase the accuracy of the measurement, it is necessary to extend the test frequency range.

The extra resonances identified in Figure 5.8(d) are the results of the notch reflections. The discrepancy between the FFT accelerances at R=0.5 and 0.75 implies the geometry of the defect is not uniform across the radius. However, the propagation
velocity based on the averaged resonant interval, 1123Hz, and the distance from the top of the pile to the location of the notch, 81cm, is computed to be 1819m/s which is not close the propagation velocity, 3618m/s, computed for the primary reflection waves. The possible reasons that result in this consequence have been discussed in the foregoing paragraph.

For pile A06-131, the propagation velocity of the primary reflections was computed as 3690m/s based on the averaged time interval between $P_1$'s as shown in Figure 5.9(c) and the assumption that the primary reflection source is the bottom of the pile. The amplitude of the first primary reflection $P_1$ is the same for signals measured at $R=0.5$ and 0.75 which implies the discontinuity does not affect the primary wave reflections. The existence of the secondary reflection peaks, $S_1$, gives the evidence that a reflection source other than the pile tip exists. The signals exhibit no apparent difference between $R=0.5$ and 0.75 which implies the geometry of the defect across the radius is uniform. However, based on the measured time interval of 0.37ms between the primary and secondary reflections and the propagation velocity 3690m/s, the propagation distance between the defect and the pile tip was computed to be 69cm which is not close to the expected value of 31cm for the same reason discussed.

Results of evaluation on the piles A06-101 and A06-131 suggests when the size of the defect is relatively small or relatively close to the primary reflection source, the conventional impulse response method is only capable of detecting the "existence" of a defect but cannot efficiently identify the type and exact location of the defect. This is possibly a result of the wavelengths of the hammer generated waves being large
compared to the distance between the defect and the bottom of the pile.

Figure 5.11 shows the dispersion curves characterized by a range of Poisson’s ratio from 0.16 to 0.28 superimposed by the phase velocities corresponding the measured resonant frequencies. These phase velocities were computed based on the assumption that the resonance was the result of bottom reflection. The shear wave velocities applied to convert the non-dimensional phase velocity dispersion curves are computed by the Poisson’s ratios listed in Tables 5.10 and 5.11. As can be seen, the variations between the measured data and theoretical solution are much more significant for defective piles A06-101 and A06-131 than the intact pile A06-091. Nevertheless, the “best matched” Poisson’s ratio and shear wave velocity can still be approximated by identifying the dispersion curve that best matched the measured data. Results suggest for the pile A06-101, the Poisson’s ratio and the bulk shear wave velocity are 0.28 and 2231m/s, respectively, while for the pile A06-131, the Poisson’s ratio and shear wave velocity are 0.16 and 2423m/s, respectively.

The uncertainty for determining the characteristic dispersion curve for the defective group A piles arises from the significant variation between the theoretical solutions and the experimental results. For obtaining more convincing results of interpretation, it is necessary to extend the useful frequency range of the experiment.
Figure 5.11: Best Match Approach for all Group A 152mm (6inch) diameter Piles: A06-091, A06-101 and A06-131. The Poisson’s ratio and shear wave velocity can be found by comparing the phase velocities measured from the impulse test results and the L(0,1) mode solutions computed numerically.
5.7 Experimental Results: Group B Piles

5.7.1 Traction-free Condition

Impulse response test was conducted on the prototype pile B10-222 in traction-free boundary condition at the 15th day after construction. Figure 5.12 shows the results in terms of input force versus time, responding accelerations of surface vibration versus time, input force spectrum, and the accelerance response of the pile. Apparent wave peaks as a result of bottom reflection can be identified from the time trace of the acceleration of the surface vibration. The propagation velocity is computed as 3982 m/s based on the time-domain response, and is computed as 3929 m/s based on the resonance spectrum. The maximum useful frequency is computed as 2857 Hz based on the inverse of the impact duration $\Delta t_h$. Nevertheless, the accelerance response of the pile shows the clear resonant peak are identifiable up to the frequency of 4443 Hz.

Figure 5.13 shows the amplitude of the peaks of the reflection wave versus time. Linear regression suggests the measured data has the trend

$$\ln(A) = -0.1283t + 3.6032$$

(5.23)

where $A$ is the amplitude of the wave peak in volts. As will be discussed later, the attenuation, $\xi$ in terms of neper per unit length can be computed from the coefficient of $t$ divided by the wave propagation velocity, $c$, such that

$$\xi = \frac{-0.1283}{3929} = -3.27 \cdot 10^{-2} \text{neper/m}$$

(5.24)
Figure 5.12: Pile No. B10-122, Traction-free boundary condition: Results of Impulse Response Test. (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at R=0.5 (d) The frequency responses of the pile measured at R=0.5.
The attenuation computed herein is a result of material damping and can be further normalized by multiplying the radius of the pile

\[ \Xi_i = \xi_i a = -4.14 \cdot 10^{-3} \text{neper} \]  

(5.25)

This non-dimensional attenuation will be used as a comparison baseline for evaluating the significance of material damping in the total measured attenuation for the embedded piles later in this section.

Figure 5.14 shows the phase velocities measured from the harmonic resonant frequencies and the superimposed phase velocity dispersion curves developed from a range of Poisson's ratio 0.14 to 0.28. The first to fifth harmonics of the fundamental resonance are identified at f=898Hz, 1787Hz, 2695Hz, 3574Hz, and 4443Hz such that the corresponding phase velocities can be computed to be 3987m/s, 3967m/s, 3989m/s, 3967m/s, and 39445m/s, respectively. The non-dimensional dispersion curves are converted into dimensional solutions by the shear wave velocity computed from various Poisson’s ratios and the bar wave velocity assumed to be equal to the measured propagation velocity, 3982m/s. As can be seen, the least variation was identified between the dispersion curve with Poisson’s ratio of 0.20 and the measured data. Thus, this 15th day traction-free prototype pile can be characterized by the Poisson’s ratio of 0.20 and the shear wave velocity 2570m/s.
Figure 5.13: Pile No. B10-122: Traction-free boundary condition; Attenuation

\[ \ln(A) = -0.1283 \, t + 3.6032 \]
Figure 5.14: Pile No. B10-122: Traction-free boundary condition; Finding the Poisson's ratio using the best match approach based on the assumption $c_B = c_p$. 
5.7.2 Embedded Condition

Figures 5.15 to 5.18 show the results of the impulse response tests of the embedded piles B10-222, B12-182, B12-240, and B18-173 in terms of force versus time, acceleration versus time, FFT force spectrum, and the accelerance response. The propagation velocity can be determined from the measured acceleration waveform by dividing the propagation distance, 2L, with the averaged time intervals between waveform peaks as shown in part (c) of the Figures 5.15 to 5.18. Alternatively, the propagation velocity can also be determined from the accelerance spectra of the pile by multiplying the averaged resonant interval, Δf by the known propagation distance, 2L.

Table 5.12 lists the propagation velocities computed from the averaged interval of the reflection peaks and the averaged interval of the resonant peaks. As can be seen, the propagation velocities determined from the averaged resonance interval are consistently smaller than the one determined from the averaged reflection intervals. For larger diameter piles, the difference between the propagation velocity measured from these two approaches becomes slightly greater. For larger diameter piles, the propagation velocities start to decrease at lower frequencies than for smaller diameter piles and thus cause the resonant intervals to decrease. The estimated propagation velocity is essentially the averaged velocity of the phase velocity of the frequency components in the frequency range of these resonant peaks. For the waves measured in the time domain, the waveform is controlled by the wave modes that carry most of the energy. As shown in the FFT force spectrum, most of the excited energy distributes at the lower frequency range such that the observed reflection waveforms are controlled by the lower frequency components. The theoretical L(0,1) branch phase velocity dis-
Figure 5.15: Prototype Pile No. B10-122, embedded boundary condition:
(a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at R=0.25, 0.5, and 0.75 (d) The frequency responses of the pile measured at R=0.25, 0.5, and 0.75.
Figure 5.16: Prototype Pile No. B12-182, embedded boundary condition: (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at $R=0.25, 0.5,$ and $0.75$ (d) The frequency responses of the pile measured at $R=0.25, 0.5,$ and $0.75.$
Figure 5.17: Prototype Pile No. B12-240, embedded boundary condition: (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at $R=0.25$, 0.5, and 0.75 (d) The frequency responses of the pile measured at $R=0.25$, 0.5, and 0.75.
Figure 5.18: Prototype Pile No. B18-173, embedded boundary condition: (a) Time waveform of the impact force, (b) Spectrum of the impact force, (c) Vibration responses measured at $R=0.25$, $0.5$, and $0.75$ (d) The frequency responses of the pile measured at $R=0.25$, $0.5$, and $0.75$. 
persion curve indicates that the waves propagate faster in the lower frequency range than in the higher frequency range. Therefore, the value of the propagation velocity estimated by the averaged resonant interval will be smaller than that estimated by the time-domain reflection waveforms.

<table>
<thead>
<tr>
<th>Pile</th>
<th>Propagation Velocity (m/s), c</th>
<th>Error(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Time-domain Waveform</td>
<td>Resonance Spectrum</td>
</tr>
<tr>
<td>B10-222</td>
<td>4150</td>
<td>4094</td>
</tr>
<tr>
<td>B12-182</td>
<td>3913</td>
<td>3845</td>
</tr>
<tr>
<td>B12-240</td>
<td>4097</td>
<td>4034</td>
</tr>
<tr>
<td>B12-173</td>
<td>3827</td>
<td>3739</td>
</tr>
</tbody>
</table>

Mode Shapes

Figure 5.19 shows the numerically derived displacement profiles for the L(0,1) modes for \( \Omega \) varying from 0.25 to 2.00. A value of \( \Omega \) of 2 represents the approximate maximum useful frequency range that can be excited by the hammer used for these tests. The measured vibration is essentially the superimposed effect of all the excited wave modes in the useful frequency range. As can be seen, the displacement of each mode evaluated in the figure is uniformly distributed across the radius of the pile. The vibration amplitude decreases with increasing frequency as exhibited from Figures 5.19(a) to (h) which implies the mode shapes of lower frequency components of L(0,1) mode
will dominate the measured waveforms. The superimposed vibration is expected to be uniformly distributed across the pile.

The displacement distribution of the surface vibration is evaluated by twice integrating the averaged acceleration waveforms measured at $R=0.25$, 0.5, and 0.75 on the top of the pile. Intuitively, the numerically-derived mode shapes presented in terms of displacement can be evaluated directly by the measured acceleration because the only differences between the acceleration and displacement are the phase and amplitude. However, the hammer-generated waveform is transient rather than steady-state and contains finite length or incomplete harmonic waves which in turn may cause the spectrum to smear. Therefore, the averaged acceleration waves have to be twice integrated before conducting the mode shape evaluation. The following equations illustrate the approach to derive displacement waveform from the acceleration waveform.

\begin{align}
    c(t) &= \int_0^t a(\tau) d\tau \\
    u(t) &= \int_0^t c(\tau) d\tau
\end{align} \tag{5.26} \tag{5.27}

where $a(t), c(t), u(t)$ represent the acceleration, velocity, and displacement at the time of $t$.

Figures 5.20 to 5.23 show the measured surface accelerations in the first 10 ms of response and the computed surface velocities and displacements for piles B10-222, B12-182, B12-240, and B18-173. As can be seen, the propagation velocity recal-
Figure 5.19: The L(0,1) mode displacement profiles for Non-dimensional frequency, $\Omega$, from 0.25 to 2 ($\nu = 0.23$) are uniform across the pile.
culated based on the intervals of the reflection peaks of the converted velocity and displacement waveform for each pile is essentially the same as the one calculated from the measured acceleration waveform. This implies the acquired data do not contain significant noise or drifted signals, and show the quality of the measurements are high.

The mode shape of the waveforms excited by the hammer impact is evaluated by comparing the surface displacement vibrations between different measurement locations. The part c of Figures 5.20 to 5.23 shows that a good match on the maximum amplitude of the surface vibration displacement can be identified between the waveforms corresponding to R=0.25, 0.5, and 0.75, especially after the second or third reflection peak. The first arrival wave is usually not used for the mode analysis because it is essentially a combination of the surface wave and body wave from impact, rather than an axisymmetrical wave that has reflected from the bottom.

The agreement between the results of the impulse response tests and the numerical analysis implies that the waves generated by hammer impact are primarily the L(0,1) mode. In addition, these results suggests that for impulse response tests on cylindrical specimens, the measurement location does not have significant influence on the measurement result because of the uniformly distributed mode shape.

Attenuation

For embedded piles, the distance that an axisymmetrical wave is able to travel is controlled by the effect of the superposition of the geometrical and material attenuation. When a wave propagates along a concrete pile embedded in soil, part of the energy
Figure 5.20: Prototype Pile. B10-222, embedded boundary condition: (a) The acceleration waveform measured at R=0.25, 0.5, and 0.75, (b) The velocity trace integrated from the acceleration waveform, (c) The displacement trace integrated from the velocity waveform.
Figure 5.21: Prototype Pile No. B12-182, embedded boundary condition: (a) The acceleration waveform measured at $R=0.25$, $0.5$, and $0.75$, (b) The velocity trace integrated from the acceleration waveform, (c) The displacement trace integrated from the velocity waveform.
Figure 5.22: Prototype Pile No. B12-240, embedded boundary condition: (a) The acceleration waveform measured at R=0.25, 0.5, and 0.75, (b) The velocity trace integrated from the acceleration waveform, (c) The displacement trace integrated from the velocity waveform.
Figure 5.23: Prototype Pile No. B18-173, embedded boundary condition: (a) The acceleration waveform measured at $R = 0.25$, 0.5, and 0.75, (b) The velocity trace integrated from the acceleration waveform, (c) The displacement trace integrated from the velocity waveform.
carried by the propagating wave radially leaks into the surrounding soil. Theoretically, the radially lost energy can be quantitatively computed from the numerically-derived attenuation coefficient. However, in addition to the radially lost energy, the measurable attenuation also contains material damping and the energy loss caused by reflection at the pile tip. The material damping is a result of the scattering and absorption of waves at the interfaces between the aggregate and cement. The energy loss at the tip of the pile is a result that part of the wave energy is transmitted into the soil below the tip of the pile while the rest is reflected back.

Figures 5.24 to 5.27 plot the magnitude of reflection peaks in terms of acceleration, velocity, and displacement versus time for each of the group B prototype piles. A trendline derived from linear regression is superimposed on each of the plot, and is expressed in the form of:

\[ \ln(A) = \alpha t + \beta \]  

(5.28)

where \( A \) is the measured magnitude of waveform peak, and \( \alpha, \beta \) are coefficients derived from the linear regression.

The attenuation of the waveforms can be calculated by substituting the time and amplitude between two waveform peaks into Equation 5.28 based on the results of impulse response tests that the waves generated by hammer impact are the \( L(0,1) \) modes and the results of numerical analysis that the geometrical attenuation coefficient is
Figure 5.24: Prototype Pile No. B10-222: amplitude of the wave peaks versus time (semi-log scale)

Figure 5.25: Prototype Pile No. B12-182: amplitude of the wave peaks (semi-log scale) versus time
Figure 5.26: Prototype Pile No. B12-240: amplitude of the wave peaks versus time (semi-log scale)

Figure 5.27: Prototype Pile No. B18-173: amplitude of the wave peaks versus time (semi-log scale)
constant for all the wave modes within the corresponding frequency range.

\[
\ln\left(\frac{A_1}{A_2}\right) = \alpha(t_1 - t_2) \quad (5.29)
\]

\[
= \alpha \Delta t \quad (5.30)
\]

\[
= \alpha \frac{\Delta L}{c} \quad \text{(neper)} \\
\]

\[
\frac{\ln\left(\frac{A_1}{A_2}\right)}{\Delta L} = \frac{\alpha}{c} \quad \text{(neper/length)} \\
\]

where \(\Delta L\) represents the propagation distance over the time interval, \(\Delta t\), between waveform peaks. The unit of the attenuation calculated from Equation 5.32 is neper per unit length which is equal to the imaginary part of wave number derived from the numerical model for dispersion curve, and thus

\[
\frac{\alpha}{c} = \xi_i \quad \text{(neper/length)} \quad (5.33)
\]

For attenuation in terms of dB per unit length, Equations 5.32 and 5.33 can be rewritten as

\[
20 \log\left(\frac{A_1}{A_2}\right) = 8.686 \frac{\alpha}{c} \quad \text{(dB/length)} \quad (5.34)
\]

\[
= 8.686 \xi_i \quad \text{(dB/length)} \quad (5.35)
\]

The attenuation calculated by Equation 5.32 can be normalized by multiplying the pile radius, \(a\).

\[
\Xi_i = \xi_i a \quad (5.36)
\]
Table 5.13 lists the attenuation coefficients computed from the measured acceleration waveform and the converted velocity and displacement waveforms for all embedded group B prototype piles. The derivation of the geometrical attenuation coefficients as presented in the numerical approach were based on the ratio of the displacement amplitudes between two selected locations along an infinitely long waveguide whereas the experimentally measured quantity is the surface acceleration. Nevertheless, for steady-state waves, the attenuation coefficients derived from displacement and acceleration waveform are the same based on

\[
\begin{align*}
    u &= u(r)e^{i(\omega t - \xi z)} \quad (5.37) \\
    \theta &= \ln\left(\frac{u_1}{u_2}\right) \quad (5.38) \\
          &= \ln\left(\frac{\dot{u}_1}{\dot{u}_2}\right) \quad (5.39) \\
          &= \ln\left(\frac{a_1}{a_2}\right) \text{ neper} \quad (5.40)
\end{align*}
\]

where \(u\) and \(a\) represents displacement and acceleration, respectively. However, waves generated by a hammer impact are transient-state rather than steady-state in nature. Although a transient wave can be represented by an infinite number of harmonic waves, part of the harmonic waves that constitute the transient wave are not continuous. The evaluation of the measured acceleration, and the computed velocity, or displacement waveforms are conducted by comparing the attenuation coefficients for each of these waveforms.

As shown in column (8) of Table 5.13, the errors between the measured attenuation coefficient and the mean attenuation coefficient decrease for piles with larger
L/D ratio. For example, the errors for the pile B10-222 with L/D ratio of 8.46 were computed to be from -0.69% to 0.69% while the errors for the pile B18-173 with L/D ratio of 3.99 were computed to be from -19.74% to 12.35%. Results of the evaluation suggest that for piles with larger L/D ratio, the attenuation coefficient can be approximated by the directly measured accelerations. For piles with smaller L/D ratio, the increasing discrepancy between the attenuation coefficients computed directly from the measured acceleration waveform and the computed displacement waveform suggests that significant uncertainty may occur if one estimates the geometrical attenuation coefficient using the directly measured accelerations.

Figure 5.28 shows the comparison between experimentally-computed attenuation coefficients and the numerically developed attenuation dispersion curves with $\nu=0.14$ and 0.28 within the frequency range from $\Omega=0$ to $\Omega=2$. Figure 5.28(a) plots the directly-measured attenuation coefficients while Figure 5.28(b) presents the attenuation coefficients computed by subtracting a typical value of material damping coefficient $-4.14 \cdot 10^{-3}$ neper. This material damping coefficient was determined from the result of the traction-free pile B10-222 as presented in Section 5.7.1. For both cases, insignificant variations were identified between the experimentally-determined attenuation coefficients and numerically-derived values. Although the directly measured attenuation is a superimposed effect of material damping and geometrical attenuation, results of the evaluation implies the total attenuation can be approximated simply by the radially lost energy numerically-computed from the predetermined soil and pile parameters.
Table 5.13: The Attenuation Coefficient for Group B Prototype Piles

<table>
<thead>
<tr>
<th>Pile</th>
<th>Waveform</th>
<th>( c ) (m/s)</th>
<th>( \alpha ) (np/ms)</th>
<th>( 8.686\xi_i^a ) (dB/m)</th>
<th>( \xi_i^b ) (np/length)</th>
<th>mean ( \xi )</th>
<th>error (%)</th>
<th>L/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
<td>(7)</td>
<td>(8)</td>
<td>(9)</td>
</tr>
<tr>
<td>B10-222</td>
<td>acceleration</td>
<td>4150</td>
<td>-0.4966</td>
<td>-1.04</td>
<td>-1.52E-02</td>
<td>-1.52E-02</td>
<td>0.00</td>
<td>8.74</td>
</tr>
<tr>
<td></td>
<td>velocity</td>
<td>4150</td>
<td>-0.5000</td>
<td>-1.05</td>
<td>-1.53E-02</td>
<td>-1.52E-02</td>
<td>0.69</td>
<td></td>
</tr>
<tr>
<td></td>
<td>displacement</td>
<td>4179</td>
<td>-0.4966</td>
<td>-1.03</td>
<td>-1.51E-02</td>
<td>-1.52E-02</td>
<td>-0.69</td>
<td></td>
</tr>
<tr>
<td>B12-182</td>
<td>acceleration</td>
<td>3913</td>
<td>-0.6120</td>
<td>-1.36</td>
<td>-2.38E-02</td>
<td>-2.17E-02</td>
<td>9.71</td>
<td>5.96</td>
</tr>
<tr>
<td></td>
<td>velocity</td>
<td>3823</td>
<td>-0.5260</td>
<td>-1.17</td>
<td>-2.10E-02</td>
<td></td>
<td>-3.47</td>
<td></td>
</tr>
<tr>
<td></td>
<td>displacement</td>
<td>3827</td>
<td>-0.5115</td>
<td>-1.16</td>
<td>-2.04E-02</td>
<td></td>
<td>-6.24</td>
<td></td>
</tr>
<tr>
<td>B12-240</td>
<td>acceleration</td>
<td>4097</td>
<td>-0.4746</td>
<td>-1.01</td>
<td>-1.77E-02</td>
<td>-1.80E-02</td>
<td>1.98</td>
<td>7.87</td>
</tr>
<tr>
<td></td>
<td>velocity</td>
<td>4068</td>
<td>-0.4880</td>
<td>-1.04</td>
<td>-1.83E-02</td>
<td></td>
<td>1.51</td>
<td></td>
</tr>
<tr>
<td></td>
<td>displacement</td>
<td>4174</td>
<td>-0.4956</td>
<td>-1.03</td>
<td>-1.81E-02</td>
<td></td>
<td>0.47</td>
<td></td>
</tr>
<tr>
<td>B18-173</td>
<td>acceleration</td>
<td>3827</td>
<td>-0.3950</td>
<td>-8.97</td>
<td>-2.36E-02</td>
<td>-2.20E-02</td>
<td>7.39</td>
<td>3.99</td>
</tr>
<tr>
<td></td>
<td>velocity</td>
<td>3823</td>
<td>-0.4128</td>
<td>-9.38</td>
<td>-2.47E-02</td>
<td></td>
<td>12.35</td>
<td></td>
</tr>
<tr>
<td></td>
<td>displacement</td>
<td>3827</td>
<td>-0.2952</td>
<td>-6.70</td>
<td>-1.76E-02</td>
<td></td>
<td>-19.74</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\) attenuation = 8.686\xi_i = 8.686 \times \frac{\alpha}{c}\n
\(^b\) non-dimensional attenuation = \xi_i a = \frac{\alpha}{c} \times a
Dispersion Curves

Theoretically, the measured phase velocities will disperse as computed from a phase velocity dispersion curve that can be numerically developed from the elastic constants of the pile and the surrounding soils. If the phase velocity dispersion curve of a pile can be determined experimentally from the impulse response results, the wave propagation characteristics at higher frequency can be numerically forecasted. However, to identify the characteristic dispersion curve for the pile being tested is not a straightforward task because of the uncertainty in determining the Poisson's ratio from concrete specimens. An experimental approach for identifying the characteristic dispersion curve and shear wave velocity for a pile is described hereafter.

First, compute a series of non-dimensional phase velocity dispersion curves based on the elastic constants of the pile and the surrounding soils. For the embedded piles tested herein, a following parameters were used: a shear modulus ratio of 740, the density ratio of 1.4, a surrounding soil's Poisson's ratio of 0.3, and a reasonable range of Poisson's ratio for concrete from 0.14 to 0.28. Second, by assuming the measured propagation velocity equals the bar wave velocity, estimate the bulk shear wave velocity from Equation 5.16. Third, use this computed bulk shear wave velocity to convert the non-dimensional phase velocity dispersion curves corresponding to each Poisson's ratio into dimensional solutions, as was presented in Section 5.5. Finally, superimpose the measured phase velocities on the computed dispersion curves, and identify the computed dispersion curve that best matches the measured data. The identified dispersion curve is characterized by the Poisson's ratio of the pile and the bulk shear wave velocity, and can be used as input for higher frequency guided wave
Figure 5.28: The comparison between the numerically-derived attenuation dispersion curves and the mean of the measured attenuation coefficients for all group B piles.
analyses.

Figure 5.29 illustrates the measured phase velocities and the superimposed dispersion curves for each group B pile corresponding to Poisson's ratio from 0.14 to 0.18. These dispersion curves are converted from the non-dimensional dispersion curve based on the assumption that the bar wave velocity is equal to the propagation velocity measured from the time-domain waveform listed in Table 5.12. The "best match" approach was performed by comparing the measured phase velocities and the converted dispersion curves and identifying the best matched numerical solutions for the experimental data. The comparison shows the shear wave velocities are 2625 m/s, 2554 m/s, and 2592 m/s for piles B10-222, B12-182, and B12-240, respectively. The dispersion curves are characterized by the Poisson's ratio of 0.25, 0.20, and 0.28 for piles B10-222, B12-182, and B12-240, respectively.
Figure 5.29: Group B Prototype Piles: the measured phase velocities and the theoretical dispersion curves
As can be seen, a characteristic phase velocity dispersion curve can be identified for the piles with a large $L/D$ ratio of about 6 or more, for instance, piles B10-222, B12-182, and B12-240. This implies the assumption that the propagation velocity is equal to the bar wave velocity is valid. However, for pile B18-173 with a small $L/D$ ratio of 3.99, no apparent match can be identified between the measured data and the converted dispersion curves. The value of the measured propagation velocity is essentially a weighted average of all the wave components excited in the hammer-impact generated frequency range. The phase velocities start to drop at lower frequencies for large diameter piles than smaller diameter piles. As a result, the propagation velocity measured from the time-domain waveform is much smaller than the bar wave velocity for these piles.

For piles with small $L/D$ ratio, the characteristic phase velocity dispersion curve cannot be identified by the approach assuming the bar wave velocity equals to the propagation velocity. Nevertheless, if the measured phase velocities exhibit an apparent trend of dispersion, the characteristic dispersion curve of the pile being tested can be determined by the trial and error approach described hereafter.

Assume the shear wave velocity for pile B18-173 is 2450m/s, 2500m/s, 2525m/s, 2550m/s, or 2600m/s, and normalize the measured phase velocity by these assumed shear wave velocities. Figure 5.30 shows the normalized data superimposed by the numerically-derived non-dimensional dispersion curves. As can be seen, the best-matched numerical solution and normalized experimental data are identified between the dispersion curve with $\nu=0.25$ and the measured phase velocities normalized from
$c_T = 2500 \text{m/s}$.

Table 5.14 lists the Poisson's ratio of the characteristic dispersion curve and the bulk shear wave velocity for each group B pile. The Poisson's ratio determined from the results of impulse response tests can be used to derive higher branches of dispersion curves. The bulk shear wave velocity is a conversion factor for normalizing the measured data or converting the non-dimensional dispersion curve into the dimensional solution. If a guided wave test is conducted at a higher frequency range, the measured propagation velocity and the theoretical solutions can be compared on the same basis by applying the Poisson's ratio and shear wave velocity determined from the approach introduced herein.

<table>
<thead>
<tr>
<th>Pile</th>
<th>$c_B$ (m/s)</th>
<th>Poisson's Ratio</th>
<th>$c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B10-222</td>
<td>4150</td>
<td>0.25</td>
<td>2625</td>
</tr>
<tr>
<td>B12-182</td>
<td>3957</td>
<td>0.20</td>
<td>2554</td>
</tr>
<tr>
<td>B12-240</td>
<td>4147</td>
<td>0.28</td>
<td>2592</td>
</tr>
<tr>
<td>B18-173*</td>
<td>3827</td>
<td>0.25</td>
<td>2500</td>
</tr>
</tbody>
</table>

*Characteristic dispersion curve not identifiable by the approach based on the assumption that the bar velocity equals to the propagation velocity
Figure 5.30: Identifying the best-matched dispersion curve for B18-173 by comparing the L(0,1) branch non-dimensional phase velocity dispersion curves with the measured phase velocity normalized by selected shear wave velocities.
Application of the universal mode

The bulk shear wave velocity is the key factor for converting between non-dimensional and dimensional solutions. The bulk shear wave velocities listed in Table 5.14 were determined from assuming $c_B = c_p$ or from the trial and error approach. The relationship between a characteristic dispersion curve and measured data is very sensitive to small changes in the value of phase velocity. As presented in Chapter 3, a universal mode which is independent of the Poisson’s ratio and the other elastic constants of the pile and the soil can be identified at each branch of the dispersion curves. The universal mode frequency (UMF) for the $L(0,1)$ branch has been identified to be $\Omega = 2.6$. If the frequency of the universal mode can be experimentally identified, the shear wave velocity corresponding to this mode will be determined without the uncertainties associated with Poisson’s ratio as discussed in the foregoing section. The feasibility of determining the shear wave velocity based on the universal mode frequency is evaluated using the results of impulse response tests of group B piles.

If there exist a resonance corresponding to the universal mode, the resonant frequency is expected to be observed at $f=8553\,\text{Hz}$, $6934\,\text{Hz}$, $7038\,\text{Hz}$, and $4525\,\text{Hz}$ for the piles B10-222, B12-182, B12-240, and B18-173, respectively, based on the UMF of 2.6. The expected UMFs are computed by Equation 5.41 using on the bulk shear wave velocity listed in Table 5.14.

$$f = c_T \cdot \frac{2.6}{2\pi a}$$  \hspace{1cm} (5.41)

Figures 5.31 and 5.32 show the force spectrum, acceleration spectrum, frequency
response spectrum, and the location of the expected UMF for the piles B10-222, B12-182, B12-240, and B18-173. Whether or not a universal mode corresponded resonance exists is indicated on these figures. As can be seen in the acceleration spectrum, except for B18-173, a resonant peak can be identified near the expected UMF at 8904Hz, 6951Hz, 7382Hz for B10-222, B12-182, and B12-240, respectively. Assuming these resonant peaks are caused by the universal mode resonance, the bulk shear wave velocity can be computed by

$$c_T = \frac{2\pi UMF_a}{2.6}$$  \hspace{1cm} (5.42)

For the pile B18-173, however, the two closest resonant peaks are the fourth and fifth harmonics of the fundamental resonance of the pile. As can be seen, the spectrum energy is smeared between these two resonant peaks such that the possibly existing UMF corresponded resonant peak can not be observed. Table 5.15 lists the computed bulk shear wave velocity.

<table>
<thead>
<tr>
<th>Pile</th>
<th>Expected UMF (Hz)</th>
<th>The Closed Resonant Peak (Hz)</th>
<th>$c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B10-222</td>
<td>8553</td>
<td>8904</td>
<td>2733</td>
</tr>
<tr>
<td>B12-182</td>
<td>6934</td>
<td>6951</td>
<td>2560</td>
</tr>
<tr>
<td>B12-240</td>
<td>7038</td>
<td>7382</td>
<td>2719</td>
</tr>
</tbody>
</table>

Table 5.15: The Bulk Shear Wave Velocity Computed from the Universal Mode Resonant Frequency for Group B Piles

Potential difficulties for evaluating these particular cases based on the results of
Figure 5.31: Identification of the universal mode resonance for piles B12-182 and B12-240: A resonant peak can be identified at a frequency close to the expected UMF in both the spectra of the responding vibration and the frequency response of the pile.
Figure 5.32: Identification of the universal mode resonance for piles B10-222 and B18-173: A resonant peak can be identified at a frequency close to the expected UMF in both the spectra of the responding vibration and the frequency response of the pile.
impulse response tests include

1. The expected universal mode frequency (UMF) is higher than the maximum useful frequency, $\frac{1}{\Delta t}$, generated by the hammer impact.

2. The expected UMFs are near the null force frequency or in the frequency range where the energy is insufficient to excite the propagating waves.

As a result, the signal beyond the maximum useful frequency is difficult to interpret because the frequency response of the pile determined from dividing the measured acceleration by the insignificant input force will contain significant noise. As can be seen, a spurious or "ghost" resonant frequency in the frequency response spectrum is a result of the near zero value input force rather than a real structural resonance. The measured acceleration in this frequency range is also affected by the input force, for example, the resonance condition is not clear when it is near to the null force frequency.

Results of the impulse response tests on the pile B12-182 summarized in Table 5.16 show the values of $c_T$ and $\nu$ yield by the $c_B = c_p$ approach and the UMF approach are essentially the same when the null force frequency of the hammer-generated input force is higher than the measured UMF. The null force frequency represents the upper limit of the theoretical maximum frequency. If one uses a hammer that can induce higher frequencies than that expected for UMF, then the UMF approach will yield valid results.

Figure 5.33 plots the normalized phase velocities based on the shear wave de-
Table 5.16: Shear Wave Velocity and Poisson’s Ratio Determined from the Results of Impulse Response Tests on Pile A12-182

<table>
<thead>
<tr>
<th>Approach</th>
<th>( c_T ) (m/s)</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_B = c_p )</td>
<td>2554</td>
<td>0.2</td>
</tr>
<tr>
<td>UMF(^a)</td>
<td>2560</td>
<td>0.2</td>
</tr>
</tbody>
</table>

\(^a\)The measured UMF=6951Hz; The measured null force frequency=7311Hz

determined from the UMF approach and the assumption that \( c_B \) equals to \( c_p \). The comparison with the numerically-derived non-dimensional phase velocity dispersion curves suggests that the characteristic dispersion curves determined from the UMF approach for piles B10-222 and B12-240 have the Poisson’s ratio of 0.14 and 0.16, respectively. Theoretically, the shear wave velocity and Poisson’s ratio determined from the UMF approach will be more accurate than the one estimated by \( c_B = c_p \) approach because the assumption that \( c_B = c_p \) essentially underestimate the bar wave velocity while the UMF is independent of all the pile and soil parameters. The shear wave velocity computed from the bar wave velocity as presented for the group B piles will be underestimated which results in the values of the normalized phase velocities being overestimated. It explains the normalized phase velocities determined from the \( c_B = c_p \) approach are consistently having higher values than the one determined from the UMF approach as illustrated in Figure 5.33.
Figure 5.33: The comparison between the phase velocities normalized by the $c_T$ determined from the assumption $c_B = c_p$ and the UMF approach.
5.8 Summary

1. Evidence of the guided wave theory in conventional impulse response testing in the prototype piles can be observed in the dispersion relations expressed as propagation velocity versus frequency, displacement distribution across the surface of the pile, and geometrical attenuation coefficients.

2. To evaluate the experimental results, the parameters of the piles and the surrounding soils, including $\frac{\mu_p}{\mu_s}$, $\frac{\rho_p}{\rho_s}$, $\nu_p$, and $\nu_s$, have to be experimentally determined, or reasonably estimated, before developing the theoretical dispersion curves.

3. Results of the evaluation indicate the dynamic Poisson's ratio is very sensitive to small changes in material properties. Because the accurate determination of the Poisson's ratio is of practical importance for guided wave evaluations, the value of the Poisson's ratio should be determined directly from the piles being tested.

4. Three approaches based on the comparison between the measured data and the numerically-derived dispersion curves were proposed for identifying the Poisson's ratio of a pile. The shear wave velocity can be computed by Equation 5.16 based on the assumption that the bar wave velocity is equal to the measured propagation velocity. One must then convert the non-dimensional phase velocity dispersion curves into dimensional solutions by applying the shear wave velocities computed from a reasonable range of Poisson's ratios. Superimpose the measured phase velocities on the same plot as the converted dispersion curves. The dispersion curve that best fit the measured data will be identified.
as the characteristic dispersion curve for the test pile. For piles with L/D ratio of 4, B18-173, the measured data did not follow a numerical curve. The Poisson's ratio of the testing pile can be identified by comparing the measured data with dispersion curves converted from the shear wave velocity determined by trial and error approach. The characteristic dispersion curve will than be the one with the best fit to the measured data. The third approach is called the UMF approach. By identifying an UMF-related resonant frequency, and determining the shear wave velocity using the universal frequency, the measured phase velocities can thus be normalized and compared with the non-dimensional dispersion curves. The dispersion curve that best matches the measured phase velocity is identified as the characteristic dispersion curve for the testing pile.

5. The shear wave velocities and Poisson's ratios determined from the “best match” approach for group A piles are summarized in Table 5.17.

Table 5.17: The Bulk Shear Wave Velocity and Poisson’s ratio Determined from the $c_B = c_P$ approach for Group A Piles

<table>
<thead>
<tr>
<th>Pile</th>
<th>Poisson's ratio, $\nu$</th>
<th>Shear wave velocity, $c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>0.16</td>
<td>2342</td>
</tr>
<tr>
<td>A08-122</td>
<td>0.15</td>
<td>2635</td>
</tr>
<tr>
<td>A06-101</td>
<td>0.28</td>
<td>2231</td>
</tr>
<tr>
<td>A06-131</td>
<td>0.16</td>
<td>2423</td>
</tr>
</tbody>
</table>

6. Results of the evaluation on the defective group A piles indicate the impulse response method cannot detect the type and location of the defect at a small
distance above the bottom of the pile. It shows the need to probe the pile with a higher frequency to identify these types of defects. In this case, the theoretical guided wave approach must be used to account for the dispersion at the higher frequencies.

7. The shear wave velocities and Poisson’s ratios for group B piles are summarized in Table 5.18. These two approaches yield the same results when the UMF is smaller than the null force frequency as the results for pile A12-182. Direct

Table 5.18: The Bulk Shear Wave Velocity and Poisson’s ratio Determined from the $c_B = c_p$ approach and UMF approach

<table>
<thead>
<tr>
<th>Pile</th>
<th>$c_B = c_p$ Approach</th>
<th>UMF Approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nu$</td>
<td>$c_T$ (m/s)</td>
</tr>
<tr>
<td>B10-222</td>
<td>0.25</td>
<td>2625</td>
</tr>
<tr>
<td>B12-182</td>
<td>0.20</td>
<td>2554</td>
</tr>
<tr>
<td>B12-240</td>
<td>0.28</td>
<td>2592</td>
</tr>
<tr>
<td>B18-173</td>
<td>0.25</td>
<td>2500</td>
</tr>
</tbody>
</table>

$^a$NI: not identifiable

evidence of the existence of the UMF mode resonance is inexplicit only based on the results of impulse response test because the maximum useful frequency of the hammer impact generated waves restricts the analysis to relatively a low frequency range. It is necessary to increase the useful frequency range so that the result of the evaluation can be more explicit.

8. The displacement profile of the surface vibration in the frequency range excited
by a hammer impact is uniformly distributed across the radius as forecasted by the numerical model. Results of this evaluation help verify the validity of the forecast model for applications in this frequency range.

9. Results of the attenuation evaluation in Section 5.7.1 and 5.7.2 suggest the total energy loss can be approximated by the geometrical attenuation coefficient and the material damping coefficient. Because the geometrical attenuation coefficient can be derived numerically with given pile and soil parameters, the penetration capability of a hammer-generated stress wave can be predicted.
Chapter 6

Frequency-Controlled Method

6.1 Introduction

The frequency content of the hammer-impact generated stress waves generally range from DC to several thousand Hertz. Evaluation shows the excited waves are primarily large wavelength $L(0,1)$ modes which lack the capability to detect small defects. Results from guided wave theory indicate that at higher frequencies, waves propagate with various mode shapes. Furthermore, within certain frequency ranges, some wave modes are able to propagate while the other modes are evanescent. These properties potentially can be applied to integrity evaluation of deep foundations. To evaluate wave propagation within the higher frequency range, it is necessary to develop a method in which the frequency content of the induced energy can be selectively input.

Theoretically, a propagating wave carrying frequency components in a narrow band can be excited by a transient burst composed of a finite number of sine wave oscillations. As will be shown, the central frequency of the excited waveforms is controlled by the cycle frequency of the sine waves. The bandwidth of the spectrum is
controlled by the length of burst.

The frequency controlled method must be able to generate a propagating wave with a mode attribute that can be forecast reliably. The forecast technique is established based on the numerical evaluation of attenuation coefficients and mode shapes. To verify the effectiveness of the mode forecast, one must develop a technique to identify the mode attribute of the acquired waveforms. This proposed technique involves waveform transformation, normalization, and the comparison with numerical results. The waveform transformation is conducted by the Fast Fourier Transform (FFT) and the Short Time Fourier Transform (STFT) such that the frequency content of the measured waveform can be analyzed in the time domain, frequency domain, or joint time-frequency domain. The measured propagation velocities are further normalized so that the comparisons with the numerically derived non-dimensional results easily can be made. The normalization is performed by dividing the measured velocity and frequency by the shear wave velocity. The shear wave velocity can be estimated by either the approach introduced in Section 5.6.1 or the UMF approach as will be presented in Section 6.4.

A new approach for evaluating pile integrity is proposed based on the frequency-controlled wave excitation method, numerical mode shape evaluation, mode identification technique, and shear wave measurement method. As will be subsequently demonstrated, this approach is able to identify smaller defects that cannot be identified with conventional impulse response tests and potentially can be used to detect a small reflection source located near the primary reflection source.
6.2 Frequency-Controlled Wave Excitation

The frequency content of the propagation wave can be controlled by selecting an appropriate input waveform. Figure 6.1 shows the schematic of the input voltage that is amplified to form a square wave. The resulting acceleration is measured in the impedance head of the vibration shaker. Each square wave cycle is composed of two spikes with opposite sign. The frequency content for an input waveform can be achieved by controlling the cycle frequency, $f_c$, and the number of the square waves. The central frequency of the input waveform is essentially the same as the $f_c$ while the bandwidth of the spectrum becomes narrower as the number of the square waves increases.

6.2.1 Numerical Evaluation of the Input Waveform

The spectrum of an input waveform can be numerically derived by applying the FFT algorithm. An example is illustrated in Figure 6.2 which shows the spectrum corresponding to 1, 2, 4, and 8 square waves with a central frequency $f_c = 25$ kHz. The density of the energy is concentrated in the central frequency range. As the number of input square wave increases, the energy density of the central frequency spectrum increases as well. A propagating wave carrying components within the narrow frequency band can thus be generated. The bandwidth of the central frequency is defined as the frequency range in which the amplitudes of the frequency components are greater than a selected value, for example, 3dB and 10dB as shown in Figure 6.3. These bandwidths can be measured from the numerically-derived spectrum and further normalized by Equation 6.1 to make comparisons between different central
Figure 6.1: Illustration of the input waveform: (a) The composition of an input waveform composed of 4 square waves, (b) The response of a 4 square wave input measured at the impedance head of the WR-F7 vibration shaker.
frequencies.

\[
\bar{\beta} = \frac{\beta}{f_c}
\]  
(6.1)

In Equation 6.1, \( \beta \) is the frequency bandwidth measured from FFT numerical spectrum and \( \bar{\beta} \) is the corresponding normalized bandwidth. Figure 6.4 shows the comparisons between 3 and 10dB bandwidths excited by 1, 2, 4, 8, and 16 of input square waves. Results suggest that one should input at least 8 square waves to minimize the energy associated with frequencies other than the central frequency. However, if the length of a pile is too short, the longer input signal may mask reflected waves.

As indicated in Figure 6.2, the spectrum of the input waveform is essentially distributed over a wider frequency range in the form of side frequency bands. The side bands are segregated from each other by a near zero amplitude frequency component. These near zero amplitude frequency components can be identified at the frequencies, \( \omega \):

\[
\omega = \frac{n}{k} f_c
\]  
(6.2)

where \( n \) is a non-negative integer, \( k \) is the number of input square waves, and \( k \) is not equal to \( n \). The primary problem caused by the side band is that spurious resonant conditions may be excited if they are at frequencies close to \( \omega \). A "ghost" resonant frequency would occur in the frequency response spectrum (mobility, accelerance, or receptance) when any signal noise in the output at \( \omega \) is divided by the near zero amplitude. As a result, methods of analyses that considered derived quantities, such as mobility based on the frequency response of the structure may not be applicable at these frequencies when the experiment is conducted by the frequency-controlled
Figure 6.2: Numerical Evaluation of the Number of Square Waves
Figure 6.3: The Definition of -3 dB and -10 dB Bandwidth
Figure 6.4: Normalized Bandwidth vs Number of Input Waves
method.

In summary, numerical results indicate that a frequency-controlled propagating wave can be excited by a transient wave composed of a finite length of square waves. The bandwidth of the input waveform is controlled by the number of input waves while the center frequency of the spectrum is controlled by the cycle frequency of the square wave. Spurious resonant conditions at some frequencies may occur because of the discontinuity of the side frequency band in the spectrum. Special care needs to be taken when using the resonant components for data interpretation.

6.2.2 Experimental Evaluation of the Input Waveform and the Performance of the WR-F7 Shaker

Longitudinal waves are excited by introducing the vibration into the pile axisymmetrically from the pile head. In this work, a vertical piezoelectric shaker, model WR-F7, is mounted at the center of the pile head. The performance of the shaker is monitored by the force or acceleration measured in the embedded impedance head above the driving point between the mounting stud and the shaker. The shaker is operated by an amplified voltage signal equivalent to the input waveform which is transmitted from the virtual waveform generator in the PC-Control System described in Chapter 4. Figure 6.5 illustrates examples of vibration excitation on prototype pile A08-122. The acceleration responses are measured at the impedance head. The measured responses are essentially the reacting acceleration between the shaker’s base and the mounting stud rather than between the shaker and the pile. The input waveforms are composed of 1, 4, and 8 square waves with a central frequency of $f_c=25$ kHz. The
sampling rate is 250kHz and the data length is of 16384 ($2^{14}$) points. Parts (a), (b), and (c) of Figure 6.5 show the acquired signals which have been averaged 20 times to reduce the level of incoherent noise. Parts (d), (e), and (f) of Figure 6.5 show the spectra of the acquired waveforms. These figures show the density of the frequency components near the central frequency increases with the increasing number of input square waves. This observation is consistent with the result of numerical evaluation in Section 6.2.1. As the number input square wave increases, the energy density of the central frequency spectrum increases as well.

Figure 6.5 (d) and (e) also show that for 1 and 4 square wave input, there are 4 significant resonances at frequencies, $P_1 = 14200$Hz, $P_2 = 22500$Hz, $P_3 = 28750$, and $P_4 = 34500$. When the number of input waveforms increases to 8, the resonant peak $P_2$ becomes insignificant because it is essentially masked by the energy of the spectrum associated with the central frequency band. Nevertheless, the other resonant peaks $P_1$, $P_2$, and $P_3$ are still significant. The source of these resonant peaks is the system’s natural frequency. When the input waveform is excited at a cycle frequency near these resonances, a better signal to noise ratio (SNR) and a waveform with a narrower frequency band can be generated.

Figure 6.6 shows the comparison of the spectra between the input waveforms having cycle frequencies of 10kHz which is randomly selected and 15kHz which is approximately equal to $P_1$. Both of the input waveforms are composed of 8 square waves and are measured at the impedance head of the vibration shaker. The excited waveforms are measured at $R=0.50$ and $0.75$ at the same surface as the shaker. The
Figure 6.5: The time domain waveform and the corresponding spectra of the 1, 4, and 8 square wave input. Evaluations are conducted on A08-122. These accelerations are measured at the impedance head. The central frequency, fc, is 25 kHz.
objective of this comparison is to demonstrate that a better quality waveform in terms of frequency content and band width can be excited if the input cycle frequency is close to one of the system's resonant frequencies, in this case $P_1 (=14200\text{Hz})$ than at a randomly selected frequency. Figures 6.6 (a) and (d) show the acceleration spectra of the input waveforms measured at the impedance head. Figures 6.6 (b) and (c) show the acceleration spectra of the excited acceleration waveforms for input $f_c=15\text{kHz}$ measured at $R=0.50$ and 0.75, respectively. Figures 6.6 (e) and (f) show the spectra of the acceleration waveform for input $f_c=10\text{kHz}$ measured at $R=0.50$ and 0.75, respectively. As can be seen in Figure 6.6(a), (b), and (c), the density of the input and responding spectra are clearly concentrated near the central frequency $f_c=15\text{kHz}$ whereas in Figure 6.6(d), (e), and (f), no apparent spectrum concentration can be observed near the frequency $f_c=10\text{kHz}$. This comparison implies that a propagating wave with a more concentrated frequency content and better signal to noise ratio can be expected when the waveform is excited with the central frequency close to $P_1$.

The performance of the WR-F7 shaker at frequencies of $22.5\text{kHz}$ and $27.5\text{kHz}$ which are equal to $P_2 (=22.5\text{kHz})$ and close to $P_3 (=28750\text{kHz})$ is evaluated by comparing the waveforms excited at $P_2$ and $P_3$ with a randomly selected cycle frequency of $f_c=25\text{kHz}$. The evaluation is conducted by the input waveforms composed of 1, 4, and 8 square waves. Figures 6.7(a), (b), and (c) show the spectra of the excited waveforms measured at the impedance head for the input cycle frequencies of $f_c=22.5\text{kHz}$, $f_c=25\text{kHz}$, and $f_c=27.5\text{kHz}$, respectively.

The evaluation shows that when the input waveform is composed of either 1 or 4
Figure 6.6: The Comparison between the Spectra of 8-Square Wave Input for $f_c=15$ kHz and 10 kHz; Testing Pile: A08-122
square waves, broad band spectra are excited regardless of the cycle frequency of the input waveform. Apparent concentration of spectrum energy as a result of the central frequency of the input waveform cannot be identified. Nevertheless, the resonant peaks at $P_1$, $P_2$, $P_3$, and $P_4$ are all identifiable.

However, when the input waveform is composed of 8 square waves, the energy of the spectrum is significantly concentrated in a frequency band whose center is at the input central frequency. This implies a narrow frequency band wave propagation can be excited by the input waveform containing at least 8 square waves. Nevertheless, the comparison of the excited bandwidth as illustrated by B.W.8 in Figure 6.7(a), (b), and (c) shows that the bandwidths excited by input waves centered at $f_c=22.5\text{kHz}$ and $f_c=27.5\text{kHz}$ are narrower than the one excited by $f_c=25\text{kHz}$. This implies that a better control of frequency content can be achieved if the waveforms are excited at or close to the resonant frequencies, in this case $P_2$ and $P_3$.

The performance evaluation for the particular WR-F7 piezoelectric shaker shows that a propagating wave carrying a narrow band frequency components centered at a target frequency can be excited by inputting a waveform composed of at least 8 square waves whose cycle frequency is equal to the target frequency. The excited waveform contains narrower bands of frequency components when the input cycle frequency is selected near one of the resonant frequencies of the system.
Figure 6.7: The spectra and 10dB bandwidth of 1, 4, and 8 square wave input at the central frequency, (a)$f_c=P_2=22.5$ kHz, (b)$f_c=25$ kHz, and (c)$f_c=P_3=27.5$ kHz.
6.3 Dispersion Curves and the Universal Mode

6.3.1 Derivation of Non-dimensional Dispersion Curves

The L(0,1) branch dispersion curves for A-group and B-group piles have been presented in Section 5.4. The frequency-controlled method extends the range of frequencies that can be applied to a pile to those where the L(0,2) and L(0,3) modes exist. Therefore, the L(0,2) and L(0,3) branches of the dispersion curves need to be developed as a baseline for data analysis. Table 6.1 recaps the input parameters used to derive the non-dimensional dispersion curves. These parameters are the same as those in Section 5.2 and 5.3.

<table>
<thead>
<tr>
<th>Pile Group</th>
<th>B-Group Pile</th>
<th>A-Group Pile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary Condition</td>
<td>Embedded</td>
<td>Traction-free</td>
</tr>
<tr>
<td>Density ratio $\left(\frac{\rho_p}{\rho_s}\right)$</td>
<td>1.5</td>
<td>$^a$2400</td>
</tr>
<tr>
<td>Shear modulus ratio $\left(\frac{\mu_p}{\mu_s}\right)$</td>
<td>740</td>
<td>$^{10^7}$</td>
</tr>
<tr>
<td>Poisson’s ratio of the pile, $\nu_p$</td>
<td>$0.14 - 0.28$</td>
<td></td>
</tr>
<tr>
<td>Poisson’s ratio of the soil, $\nu_s$</td>
<td>0.3</td>
<td>NA</td>
</tr>
</tbody>
</table>

$^a$density ratio of concrete to air

Figure 6.8 shows the first branches of dispersion relations in terms of the real and imaginary parts of non-dimensional wave number versus non-dimensional fre-
frequency. Figure 6.9 shows the non-dimensional group velocity dispersion curves and non-dimensional attenuation dispersion curves. The non-dimensional group velocity dispersion curves are converted from the real part of the dispersion relations. The non-dimensional attenuation dispersion curves are essentially equivalent to the imaginary part of dispersion relations (Hanifah, 1999). Each point along a dispersion curve is called a mode, which can be characterized by the corresponding propagation velocity, attenuation, and the displacement distribution. The modes having constant characteristics (non-dispersion) in a frequency range of interest are considered the better candidates for conducting a guided wave test. The lower frequency range ($\Omega < 1.4$), or frequencies which the conventional impulse response test is performed, contains the non-dispersive frequency ranges of L(0,1) branch where the propagation velocity, attenuation, and displacement distribution are constant.

6.3.2 Universal Frequency

Results of numerical evaluation showed that the non-dimensional attenuation is a function of the shear modulus ratio, density ratio, and Poisson’s ratio while the non-dimensional propagation velocities are a function of Poisson’s ratio only (Hanifah, 1999). For each branch of a non-dimensional dispersion curve, there exists at least one mode whose non-dimensional frequency is independent of all the input parameters, including density ratio, shear modulus ratio, and Poisson’s ratio. This frequency is herein called the “universal frequency.” This universal frequency is of practical importance because non-dimensional propagation velocities depend on the Poisson’s ratio, and the Poisson’s ratio of concrete can vary between 0.14 and 0.30. By identifying a universal frequency, the uncertainty in results that arises from not exactly know-
Figure 6.8: The Imaginary and Real Parts of the Non-dimensional Dispersion Curves for $\mu - ratio = 740, \rho - ratio = 1.5, \nu_s = 0.3$
Figure 6.9: The non-dimensional attenuation and group velocity dispersion curves for $\mu - ratio = 740, \rho - ratio = 1.5, \nu_s = 0.3$
ing the Poisson's ratio can be eliminated. An example is illustrated by Figure 6.10 in which the non-dimensional dispersion curves are derived from the shear modulus ratios ($\mu$-ratio) of 325 and 740, and Poisson's ration from 0.18 to 0.28. As can be seen, all the dispersion curves are the function of Poisson's ratio and the attenuation dispersion curves are further affected by the shear modulus ratio. A frequency $\Omega=2.6$, is identified as the same point on each of the L(0,1) dispersion curves, and thus is independent of the Poisson's ratio.

Impact-echo test results presented in Section 5.7 can be interpreted to show that a universal mode can be excited for piles, as will be discussed in Section 6.4. The frequency corresponding to this resonant peak is termed universal mode frequency (UMF) or universal frequency. The importance of the UMF is that the bulk shear wave velocity, $c_T$ can be estimated without assuming or measuring a value of Poisson's ratio. The estimated $c_T$ is essentially the normalization factor for the measured propagation velocities. The objective of the normalization is to have a common basis to compare numerical and experimental results. Equation 6.3 demonstrates the manner to estimate $c_T$ from the experimentally measured universal frequency (UMF):

$$
    c_T = \frac{2\pi a \cdot (UMF)}{2.60}
$$

(6.3)

where 2.60 is the non-dimensional universal frequency for the L(0,1) branch identified from numerical analysis, and UMF is the universal frequency identified from
Figure 6.10: The Universal Mode for L(0,1) Branch
experimental results. Measured velocities, $c$, and frequencies, $f$, are normalized by:

\[
\begin{align*}
C &= \frac{c}{c_T} \\
\Omega &= \frac{2\pi f a}{c_T}
\end{align*}
\]  \hspace{1cm} (6.4)

where $C$ represents the normalized non-dimensional velocity.

Table 6.2 recaps the UMF for the embedded Group-B prototype piles identified from the accelerance spectrum of impulse response test and the $c_T$ computed based on the identified UMF as given in Section 5.7.2. However, the use of frequency response spectrum is not available if the universal frequency is expected to occur at frequencies beyond those which can be excited by a hammer impact.

Table 6.2: The Identification of Universal Frequency and Shear Wave Velocity for B12-182 and B12-240 Using Impulse Response Method

<table>
<thead>
<tr>
<th>Pile</th>
<th>Diameter (mm)</th>
<th>UMF (Hz)</th>
<th>$c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B12-182</td>
<td>305</td>
<td>6951</td>
<td>2733</td>
</tr>
<tr>
<td>B12-240</td>
<td>305</td>
<td>7382</td>
<td>2719</td>
</tr>
</tbody>
</table>

### 6.4 Determining Shear Wave Velocity Using the Universal Frequency

Results of numerical analysis indicate that the non-dimensional universal frequency is invariant, and that the wave mode corresponding to it is independent of the Pois-
son's ratio. Based on this result, the bulk shear wave velocity can be determined by the universal mode frequency (UMF) without introducing the uncertainties arising from use of a value of Poisson's ratio that either is assumed or measured from a concrete specimen, as discussed in Section 5.2.3. As shown in Section 5.7, a resonant peak corresponding to the UMF can be identified in the frequency response spectrum, either in terms of mobility or accelerance, for embedded piles subject to impulse response tests. However, the identification of UMF using the impact response method is limited by the useful frequency range that a hammer impact can generate. For example, the UMF for a smaller diameter pile may occur at a frequency higher than the maximum useful frequency that can be generated by a hammer impact. This section discusses the UMF identification for the higher frequency applications using the frequency-controlled method. Due to the nature of the spectrum excited by the frequency-controlled method, the mobility or accelerance analysis is not applicable because the excited spectrum is generally not continuous over the broad frequency range being analyzed.

To identify the UMF, one assumes that there exists a trend in the change of the resonant frequency intervals, and this trend can be determined from the theoretical phase velocity dispersion curve. The UMF resonance is not one of the harmonics of the fundamental resonance such that the existence of UMF resonance disturbs the expected trend. By identifying the resonant peak that disturbs the balance of the expected trend, the UMF can be determined. The UMF is used to compute the bulk shear wave velocity which is essentially the conversion factor for the normalization of measured velocities and frequencies. The importance of the normalization is that the
experimental results can be compared directly with the numerical results.

6.4.1 Theoretical Basis of UMF Identification

Results of guided wave theory can be used to forecast the trend of resonant intervals with frequencies by considering the phase velocity dispersion curve. As will be seen subsequently, the resonant interval in a given frequency range can be constant, decreasing, or increasing with frequencies, depending on the dispersion characteristics of the phase velocity.

The phase velocity, \( c_n \), of a propagating wave at a measured resonant frequency, \( f_n \), is

\[
c_n = f_n \cdot \lambda_n
\]  
(6.5)

where \( \lambda_n \) is the wavelength corresponded to the resonant frequency. This wavelength can be determined from the principle that resonance occurs when the integer number of the half wave length equals to the length of cylindrical wave guide such that

\[
L = n \cdot \frac{\lambda_n}{2}
\]  
(6.6)

The integer number \( n \) represents the sequence of the harmonics of the fundamental resonance. Combining Equations 6.5 and 6.6, the phase velocity is

\[
c_n = \frac{2f_nL}{n}
\]  
(6.7)
Figure 6.11 illustrates the resonant intervals between \( f_{n-1} \), \( f_n \), and \( f_{n+1} \), and the corresponding phase velocity \( c_{n-1} \), \( c_n \), and \( c_{n+1} \) in the phase velocity dispersion curve. Rearranging Equation 6.7, the resonant frequency, \( f_n \), can be expressed as

\[
f_n = \frac{n \cdot c_n}{2L}
\]

(6.8)

The resonant interval between the \((n-1)\)-th and \(n\)-th resonant peaks is represented by \( \Delta f_n \) while the interval between the \(n\)-th and \((n+1)\)-th resonant peak is represented by \( \Delta f_{n+1} \)

\[
\Delta f_n = f_n - f_{n-1}
\]

(6.9)

\[
\Delta f_{n+1} = f_{n+1} - f_n
\]

(6.10)

By substituting Equation 6.8 into Equations 6.9 and 6.10, \( \Delta f_n \) and \( \Delta f_{n+1} \) can be written as

\[
\Delta f_n = \frac{1}{2L}[n(c_n - c_{n-1}) + c_{n-1}]
\]

(6.11)

\[
\Delta f_{n+1} = \frac{1}{2L}[(n + 1)(c_{n+1} - c_n) + c_n]
\]

(6.12)

The trend in the change of the resonant intervals with frequency corresponding to the phase velocity dispersion curve is evaluated by

\[
\Delta f_n - \Delta f_{n+1}
\]

(6.13)
Figure 6.11: The resonant frequencies and the corresponding theoretical phase velocities at non-dispersive region and dispersive region
In region I of Figure 6.11, the phase velocity is non-dispersive. The phase velocity of the propagation wave is essentially independent of its frequency component. The wave is propagating at a constant velocity $c_p$. For convenience, $c_p$ is replaced by $c$ herein. In the non-dispersion frequency range,

$$c = c_{n-1} = c_n = c_{n+1} = \textit{constant}$$  \hspace{1cm} (6.14)

Substituting the relationship in 6.14 into Equations 6.11 and 6.12, it leads to

$$\Delta f_n = \Delta f_{n+1}$$  \hspace{1cm} (6.15)

The significance of Equation 6.15 is that the resonant intervals are constant if the corresponding phase wave is non-dispersive.

In region II of Figure 6.11, the phase velocities are dispersive. The relative magnitudes of the phase velocities in this region are

$$c_{n-1} < c_n < c_{n+1}$$  \hspace{1cm} (6.16)

and the corresponding frequencies are in the sequence

$$f_{n-1} < f_n < f_{n+1}$$

Applying the relationship in 6.16 into 6.13, one can show that

$$\Delta f_n > \Delta f_{n+1}$$  \hspace{1cm} (6.17)
which implies the resonant intervals in region II of Figure 6.11 decrease with increasing frequency.

Thus, the trend of the change of resonant intervals can be determined from the corresponding phase velocity dispersion curve. The existence of the UMF resonance will disturb the expected resonant intervals of the spectrum because the resonance corresponding to the universal mode is independent of the fundamental resonance and its harmonics. This particular characteristic can be used to identify the universal frequency.

6.4.2 Procedure to Identifying the UMF

When the resonance corresponding to the universal mode is excited, the expected trend of the resonant intervals near the universal frequency will be different from the theoretical trend. The UMF can be identified if the resonance that interrupts the forecasted trend can be selected. The following procedure for experimentally identifying the UMF is based on this principle:

1. Estimate the frequency range within which the universal mode resonance is expected by knowing the non-dimensional universal frequency and assuming a range for the shear wave velocity:

\[
\frac{(U\Omega M) \cdot c_{T_L}}{2\pi \cdot a} < f_{expected} < \frac{(U\Omega M) \cdot c_{T_H}}{2\pi \cdot a}
\]  

(6.18)

where \(U\Omega M\) is the non-dimensional universal frequency, and \(c_{T_L}\) and \(c_{T_H}\) are the assumed lower bound and upper bound shear wave velocities, respectively.
For example, assume the shear wave velocity of a 127mm radius concrete pile varies between $c_{T_L} = 2200$ m/s and $c_{T_W} = 2800$ m/s, and the non-dimensional UMF is equal to 2.60. The frequency range within which the universal mode resonance could occur is

$$7168 \text{Hz} < f < 9123 \text{Hz}$$

(6.19)

2. Introduce a waveform into the pile using the frequency-controlled method. The input waveform is selected so that the excited wave carries a broad band of frequency components and is centered at a frequency within the frequency range determined in step 1. The acquired vibration responses are averaged and recorded in the time domain and transformed by FFT into frequency domain where the identification is performed.

3. Forecast the trend of the change of the resonant intervals from the numerically derived phase velocity dispersion curve, as discussed in Section 6.4.1.

4. Identify an anomalous resonant peak in the spectrum derived in step 2 that does not follow the expected trend determined from step 3. If no anomalous resonant peak can be identified, go back to step 2, choose a different input waveform or power level, and repeat steps 2 through 4. Assume the frequency corresponding to the anomalous resonant peak is UMF.

5. Back calculate the bulk shear wave velocity corresponding to the assumed UMF from step 4 using

$$c_T = \frac{2\pi \cdot (UMF) \cdot a}{(UM\Omega)}$$

(6.20)
where the $UM\Omega$ is the non-dimensional universal frequency.

6. Normalize the phase velocity, $c_p$, measured at each resonant frequency using the $c_T$ determined from step 5.

$$C_p = \frac{c_p}{c_T} \quad (6.21)$$

7. Plot the numerically-derived, non-dimensional phase dispersion curves on the same figure as the normalized phase velocity. The UMF is confirmed if the correlation between the numerical result and the normalized experimental result is significant, If not, go back to step 1, and repeat procedures from 1 to 7.

This UMF identification procedure is demonstrated herein by the traction-free piles A06-091 and A08-122, and the embedded pile B10-222. Table 6.3 shows the frequency range within which the universal mode can be expected for each of these piles based on the assumption that the shear wave velocity is between 2200 m/s and 2800 m/s.

Waveforms composed of either 4 or 8 square waves with center frequencies $f_c=13000$ Hz and 10000 Hz were used as the input vibration for the piles A06-091 and A08-122, respectively. The selection was essentially a trial and error approach until an expected anomalous resonant peak was excited. The vibration responses are measured on the same surface as the input excitation. The acquired data were recorded in the time domain and transformed by FFT into frequency domain for later analysis.
Table 6.3: The Expected Frequency Range within which the Universal Mode Resonance could Occur

<table>
<thead>
<tr>
<th>Pile</th>
<th>Radius (mm)</th>
<th>Assumed $c_T$ (m/s)</th>
<th>Lower Limit, $c_{TL}=2200$</th>
<th>Upper Limit, $c_{TH}=2800$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Lower Bound $f$ (Hz)</td>
<td>Upper Bound $f$ (Hz)</td>
</tr>
<tr>
<td>A06-091</td>
<td>76</td>
<td>11978</td>
<td>15245</td>
<td></td>
</tr>
<tr>
<td>A08-122</td>
<td>102</td>
<td>8925</td>
<td>11359</td>
<td></td>
</tr>
<tr>
<td>B10-222</td>
<td>127</td>
<td>7168</td>
<td>9123</td>
<td></td>
</tr>
</tbody>
</table>

Step 2 of the procedure for UMF identification suggests that a resonance corresponding to the UMF can be excited when the cycle frequency of the input waveform is within the frequency range shown in Table 6.3. To ensure the excitation of universal mode resonance, the cycle frequency of the input waveform should be selected at a frequency near the expected UMF. However, the universal mode resonance can still be excited if it is located in a side frequency band. This is evaluated by selecting the cycle frequency of 12500kHz for pile B10-222, for which the resonance corresponding to UMF is expected to occur between a frequency of $f=7168$Hz and 9123Hz as shown in Table 6.3.

Figure 6.12 shows the expected change of the resonant intervals for L(0,1) modes of propagating waves. The non-dimensional frequencies, $\Omega$, of 1.4 and 3.8 are selected based on the consideration that the phase velocities are approximately constant for frequencies smaller than $\Omega <1.4$ and greater than $\Omega >3.8$. The group velocity dis-
persion curves are independent of the value of the Poisson's ratio at both of the selected frequencies. The resonant intervals are approximately constant in the frequency ranges $\Omega < 1.4$, and $\Omega > 3.8$. For the frequency range between $\Omega = 1.4$ and $\Omega = 3.8$, the resonant interval decreases as the frequency increases. For $L(0,1)$ mode of propagation waves, the resonant interval is expected to be constant at the beginning of the spectrum, decrease in the intermediate frequency range, and becomes constant again thereafter. Figure 6.13 shows the acceleration spectrum of the waveforms measured at $R = 0.5$ for A06-091, A08-122, and B10-222, and the frequency range where the resonant intervals theoretically decrease with increasing frequency.

Anomalous resonant peaks are identified at $f = 12300\text{Hz}$, $10337\text{Hz}$, and $8904\text{Hz}$ for each of the above piles. Whether or not these resonant frequencies are anomalous can be determined by the trend of $\Delta f_n - \Delta f_{n-1}$ versus frequency. For piles A06-091 and A08-122, Figure 6.14 shows the trend of the resonant interval change in term of $\Delta f_n - \Delta f_{n-1}$ in the conditions that the anomalous frequency peaks are assumed to be the UMF (red line and circles), or to be one of the harmonics of the fundamental resonance (blue line and circles). When the anomalous resonant peak is interpreted as the UMF, the $\Delta f_n - \Delta f_{n-1}$ exhibits the following trend. At the lower frequency range, $\Delta f_n - \Delta f_{n-1}$ approximately equals to zero which implies $\Delta f$ is constant. At the intermediate frequency range, the values of $\Delta f_n - \Delta f_{n-1}$ become negative which means the resonant interval decreases with increasing frequency. At higher frequency range, the values of $\Delta f_n - \Delta f_{n-1}$ fluctuate evenly around zero which implies the resonant intervals is constant. This trend is in consistent with the expected one and thus confirms the result. If the anomalous resonant peak is interpreted as one
Figure 6.12: The expected change of resonant intervals based on the L(0,1) branch of dispersion curves.
Figure 6.13: The acceleration spectrum of the acquired waveforms for A06-091, A08-122, and B10-222. Resonant spectra were excited by waveforms with $f_c=12.5\text{kHz}$. The UMF corresponded frequencies are identified at $f=12300\text{Hz}$, $10337\text{Hz}$, and $8904\text{Hz}$ for A06-091, A08-122, and B10-222, respectively.
of the harmonics of the fundamental resonance, the change of \( \Delta f_n - \Delta f_{n-1} \) is not consistent with the expected trend because of the appearance of a significant spike having positive amplitude in the plot, as shown in blue diamonds in Figure 6.14.

### 6.4.3 Verification of Shear Wave Velocity Computation

Table 6.4 lists the calculated shear wave velocity based on the assumed UMF by Equation 6.20 for each of the piles. Tables 6.5 and 6.6 list the harmonic number for each of the resonant peaks, the phase velocity at each resonant frequency, and the corresponding normalized phase velocity for piles A06-091 and A08-122. The harmonic number, \( n \), represents the order of the harmonics of the fundamental resonance. The resonant peak corresponding to the assumed UMF is not a harmonic of the fundamental resonance. Thus, the phase velocity and wave length corresponding to the UMF resonance cannot be estimated from Equation 6.6. The confirmation of whether or not the anomalous frequency is the UMF is made by comparing the normalized phase velocities with the numerically-derived non-dimensional phase velocities. Figure 6.15 shows the normalized phased velocities and the imposed theoretical phase velocity dispersion curves with Poisson’s ratios between 0.14 and 0.28. The comparison shows good agreement between the experimental and numerical results. Therefore, the frequency of the anomalous resonance is confirmed to be the UMF and the approach for identifying the universal frequency is shown to be valid.

For embedded pile B10-222, Figure 6.13 (c) shows that an anomalous resonant peak can be identified at \( f=8904\text{Hz} \) at the superimposed acceleration spectrum of the waveforms measured at \( R=0.50 \) from both the results of impulse response method
Figure 6.14: The trends of the change of the resonant intervals for the conditions including and not including the anomalous frequency for A06-091 and A08-122.
Figure 6.15: The phase velocities normalized by the shear wave velocities determined from the UMF and imposed on the same plot as the numerically derived phase velocity dispersion curves for A06-091, A08-122, and B10-222.
Table 6.4: The Shear Wave Velocity for Pile A06-091, A08-122, and B10-222 Estimated from the Identified Anomalous Frequency

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>Non-dimensional UMF, Ω</th>
<th>Identified Anomalous Frequency, f(Hz)</th>
<th>Shear Wave Velocity $c_T$(m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>2.6</td>
<td>12300</td>
<td>2260</td>
</tr>
<tr>
<td>A08-122</td>
<td>2.6</td>
<td>10337</td>
<td>2538</td>
</tr>
<tr>
<td>B10-222</td>
<td>2.6</td>
<td>8904</td>
<td>2733</td>
</tr>
</tbody>
</table>

and frequency-content controlled method. One can directly assume this frequency is the UMF, determine the $c_T$, perform the normalization, and conduct the comparison with the numerically-derived phase velocities to confirm the assumption. Table 6.7 lists the normalized phase velocities based on the shear wave velocity of 2733 m/s determined from the assumed UMF of 8904Hz.

Figure 6.16 illustrates the comparison between the measured resonance spectrum and the numerically-derived phase velocity dispersion curves. The phase velocities determined from the resonant frequency and an estimated harmonic number are imposed on numerically-derived phase velocity dispersion curves. As can be seen, some harmonics of the fundamental resonances are missing from the excited spectrum. Nevertheless, the number of these missing harmonic resonant peaks can be determined by a trial and error approach, and are identified as corresponding to $n=6$, $n=10$, and $n=11$. This trial and error approach is illustrated in Figure 6.17 wherein the the change of the resonant intervals is evaluated based on the following assumptions. The red line and circles represent the computed frequency changes assuming
Table 6.5: The Measurement Normalization Based on the Measured Universal Mode Frequency (UMF=12300 Hz), File No: A06-091

<table>
<thead>
<tr>
<th>Harmonic number, n</th>
<th>Resonant frequency (Hz)</th>
<th>Wavelength λ(\text{m})</th>
<th>Phase Velocity \langle m/s \rangle</th>
<th>Normalized velocity \xi_\phi</th>
<th>Normalized frequency Ω</th>
<th>Wave Number \xi_\sigma</th>
<th>Wave Number \xi_\sigma^a</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)^b</td>
<td>(4)^c</td>
<td>(5)^d</td>
<td>(6)^e</td>
<td>(7)^f</td>
<td>(8)</td>
</tr>
<tr>
<td>1</td>
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<td>1.82</td>
<td>3844</td>
<td>1.70</td>
<td>0.45</td>
<td>3.45</td>
<td>0.26</td>
</tr>
<tr>
<td>2</td>
<td>3963</td>
<td>0.91</td>
<td>3606</td>
<td>1.59</td>
<td>0.84</td>
<td>6.90</td>
<td>0.53</td>
</tr>
<tr>
<td>3</td>
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<td>0.61</td>
<td>3579</td>
<td>1.58</td>
<td>1.25</td>
<td>10.36</td>
<td>0.79</td>
</tr>
<tr>
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<td>3544</td>
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<td>1.65</td>
<td>13.81</td>
<td>1.05</td>
</tr>
<tr>
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<td>9625</td>
<td>0.36</td>
<td>3504</td>
<td>1.55</td>
<td>2.03</td>
<td>17.26</td>
<td>1.32</td>
</tr>
<tr>
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<td>0.30</td>
<td>3435</td>
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<td>2.39</td>
<td>20.71</td>
<td>1.58</td>
</tr>
<tr>
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<td>12712</td>
<td>0.26</td>
<td>3305</td>
<td>1.46</td>
<td>2.69</td>
<td>24.17</td>
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</tr>
<tr>
<td>8</td>
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<td>0.23</td>
<td>3100</td>
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<td>2.88</td>
<td>27.62</td>
<td>2.10</td>
</tr>
<tr>
<td>9</td>
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<td>0.20</td>
<td>2902</td>
<td>1.28</td>
<td>3.03</td>
<td>31.07</td>
<td>2.37</td>
</tr>
<tr>
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<td>2723</td>
<td>1.20</td>
<td>3.16</td>
<td>34.52</td>
<td>2.63</td>
</tr>
<tr>
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<td>2577</td>
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<td>37.98</td>
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</tr>
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</tr>
<tr>
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<td>2394</td>
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</tr>
<tr>
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<td>51.88</td>
<td>3.95</td>
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<td>55.24</td>
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<td>58.69</td>
<td>4.47</td>
</tr>
<tr>
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</tr>
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<td>4.71</td>
<td>65.59</td>
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</tr>
<tr>
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<td>2124</td>
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<td>4.93</td>
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<tr>
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<td>5.14</td>
<td>72.50</td>
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<tr>
<td>22</td>
<td>25402</td>
<td>0.08</td>
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<td>1.93</td>
<td>5.37</td>
<td>75.95</td>
<td>5.79</td>
</tr>
</tbody>
</table>

\( a_{CT} = \frac{2\pi \cdot UMF \cdot a}{2.60} = 2260 m/s \)

\( b(3) = \frac{2L}{(1)} \)

\( c(4) = (2)-(3) \)

\( d(5) = \frac{c_T}{(4)} \)

\( e(6) = \frac{c_T}{fa} \)

\( f(7) = \frac{c_T}{(3)} \)
Table 6.6: The Measurement Normalization Based on the Measured Universal Mode Frequency (UMF=10337 Hz), File No: A08-122

<table>
<thead>
<tr>
<th>Harmonic number, n</th>
<th>Resonant frequency (Hz)</th>
<th>Wave Length $\lambda$ (m)</th>
<th>Phase Velocity (m/s)</th>
<th>Normalized frequency $\xi$</th>
<th>Wave Number $\xi_{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(2)</td>
<td>(3)$^b$</td>
<td>(4)$^c$</td>
<td>(5)$^d$</td>
<td>(6)$^e$</td>
</tr>
<tr>
<td>1</td>
<td>1600</td>
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<td>3904</td>
<td>1.54</td>
<td>0.40</td>
</tr>
<tr>
<td>2</td>
<td>3300</td>
<td>1.22</td>
<td>4026</td>
<td>1.59</td>
<td>0.83</td>
</tr>
<tr>
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<tr>
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<td>1.64</td>
</tr>
<tr>
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<td>3936</td>
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</tr>
<tr>
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<td>0.41</td>
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<td>1.52</td>
<td>2.39</td>
</tr>
<tr>
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<td>3769</td>
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<td>2.72</td>
</tr>
<tr>
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<tr>
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<td>2533</td>
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<td>4.18</td>
</tr>
</tbody>
</table>

$^a c_T = \frac{2\pi \cdot UMF \cdot a}{2.60} = 2538 m/s$

$^b (3) = \frac{2L}{(1)}$

$^c (4) = (2) \cdot (3)$

$^d (5) = \frac{(4)}{(4)}$

$^e (6) = \frac{c_T}{2\pi f a}$

$^f (7) = \frac{2\pi}{(3)}$
the resonant peak at \( f = 8904 \text{Hz} \) is the UMF. The blue line and diamonds represent the computed frequency changes assuming the resonance at \( f = 8904 \text{Hz} \) is the harmonic corresponding to \( n = 10 \) of the fundamental resonance. The green line and squares represents the computed frequency changes assuming the resonance at \( f = 8904 \text{Hz} \) is the harmonic corresponding to \( n = 11 \) of the fundamental resonance. The evaluation shows that when the anomalous resonance at \( f = 8904 \text{Hz} \) is assumed to be a harmonic of the fundamental resonance corresponding to \( n = 10 \) or \( 11 \), significant spikes with positive amplitude are observed, indicating that particular assumption is incorrect. The comparison between the normalized results and the numerical results confirms that the frequency of the anomalous resonance at the \( f = 8904 \text{Hz} \) is the UMF.

Comparisons between the normalized phase velocities and the non-dimensional dispersion curves not only confirms the validity of the UMF based shear wave velocity, but also provides a baseline for estimating Poisson ratio. The superimposed dispersion curves are developed from a range of various Poisson's ratios from 0.14 to 0.28. The Poisson's ratio of a test pile can be determined by identifying which of the dispersion curve best matches the normalized phase velocities. As demonstrated by Figure 6.17(a), for \( \Omega < 2.6 \), the experimentally-determined phase velocities consistently match the numerically-derived phase velocity dispersion curve developed from Poisson's ratio of 0.14. A slightly different value of Poisson's ratio 0.18 was identified for \( \Omega > 2.6 \) because the experimentally-determined phase velocities become better matched with the numerically-derived phase velocity dispersion curve with Poisson's ration of 0.18 after a plunge change of the velocity.
Table 6.7: Normalization Based on the Identified Universal Mode Frequency (UMF=8904 Hz), Pile No: B10-222

<table>
<thead>
<tr>
<th>Harmonic number, n (1)</th>
<th>Resonant frequency (Hz) (2)</th>
<th>Wave Length λ(m) (3)</th>
<th>Phase Velocity (m/s) (4)</th>
<th>Normalized frequency Ω (5)</th>
<th>Wave Number ξ (7)</th>
<th>ξ-a (8)</th>
</tr>
</thead>
<tbody>
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<td>1.98</td>
<td>3.89</td>
<td>31.13</td>
</tr>
<tr>
<td>23</td>
<td>13800</td>
<td>0.19</td>
<td>2664</td>
<td>1.97</td>
<td>4.03</td>
<td>32.55</td>
</tr>
<tr>
<td>24</td>
<td>14187</td>
<td>0.19</td>
<td>2625</td>
<td>1.96</td>
<td>4.14</td>
<td>33.96</td>
</tr>
</tbody>
</table>
Figure 6.16: (a) The L(0,1) branch of non-dimensional phase velocity dispersion curves derived from $\nu=0.14$ to 0.28 which in turn superimposed on the same plot of the normalized phase velocities based on $c_f=2733$. (b) The corresponding acceleration spectra excited by both impulse response method and frequency-controlled method (4 square wave; $f_c=12.5$ kHz) measured at $R=0.50$.
Figure 6.17: Pile B10-222; Embedded boundary condition. The trends of the change of the resonant intervals in the acceleration spectra for the conditions that the anomalous frequency peak is assumed to be the UMF and one of the harmonics of the fundamental resonance.
Table 6.8: The Confirmed Shear Wave Velocity and Estimated Poisson’s ratio

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>Poisson’s ratio, $\nu$</th>
<th>Shear Wave Velocity, $c_T$ (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Omega &lt; 2.6$</td>
<td>$\Omega &gt; 2.6$</td>
</tr>
<tr>
<td>A06-091</td>
<td>0.26</td>
<td>0.18</td>
</tr>
<tr>
<td>A08-122</td>
<td>0.26</td>
<td>0.28</td>
</tr>
<tr>
<td>B10-222</td>
<td>0.14</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Table 6.8 lists the shear wave velocity and the Poisson’s ratio determined from the best match approach. Because the measured resonant frequency is subject to $\pm 15$Hz error as a result of the sampling method and the phase velocity is computed by the multiplication of the measured resonant frequency and the corresponding wavelength, the phase velocity tends to have larger errors in the lower frequency than in the higher frequency range. Consequently, the Poisson’s ratios are slightly different between the lower frequency range and the higher frequency range. For pile A06-091, this difference seems more significant than the other piles. The wavelength corresponding to the resonant frequency is computed by dividing twice of the pile length by the harmonic number of the resonant peaks. For prototype pile with short length, the wavelength is relatively smaller compared the resonant frequency subject to measurement error. Thus, the error between the lower frequency range and higher frequency range is more significant.

This section introduces a new approach for estimating bulk shear wave velocity using the UMF. The UMF identification is conducted on the spectrum of the excited
waveforms based the expected resonant intervals determined from the phase velocity dispersion curves. The validity of the estimated shear wave velocity has been confirmed by comparing the consistency between the normalized phase velocities and numerically-derived dispersion curves. The advantage of estimating shear wave velocity from this approach is that the uncertainties caused by using an assumed Poisson's ratio or measuring the Poisson's ratio from concrete specimens can be avoided.

6.5 Numerical Mode Shapes

The integrity of a pile can be evaluated by comparing the experimental results and expected numerical results. Results of numerical evaluation in terms of dispersion curves and mode shapes can be developed from the predetermined soil and pile parameters and the boundary conditions. The frequency-controlled wave excitation is performed in the frequency range within which the wave modes having approximately constant propagation velocity and low attenuation. Numerical quantities such as power flux distribution and displacement mode shape can be used to evaluate the distribution of the energy carried by an excited wave mode or the relative difficulty for exciting a wave mode.

Results of numerical evaluation indicate the group and phase velocities, and the geometrical attenuation are approximately constant for a relatively small diameter pile over the useful frequency range excited by a conventional hammer impact. As illustrated in Figure 5.19, the displacement profiles the L(0,1) modes in this frequency range are approximately uniformly distributed across the radius of the pile. Because
the \( L(0,1) \) mode is the only axisymmetrical mode of propagation waves excited by hammer impact, it is reasonable to compare the experimental results directly with the numerically-derived \( L(0,1) \) modes. However, as introduced in Section 3.2.1, more than one mode of axisymmetrical waves exist at frequencies higher than the non-dimensional cutoff frequency of 2.88 for a pile with embedded boundary condition, and 3.44 for traction-free boundary conditions. This higher frequencies thus present a mode complicated situation wherein it is more difficult to determine which of modes can be excited. As will be presented subsequently, this problem is evaluated by considering the complexity of the expected mode shape and the relative magnitude between different modes.

The use of the mode shapes to forecast the wave modes that can be excited in practical application is illustrated by the numerically-derived \( L(0,1), L(0,2), \) and \( L(0,3) \) modes corresponding to the non-dimensional frequencies \( \Omega = 4.5 \) and \( \Omega = 5.3 \). In this range, as shown in Figure 6.9, the non-dimensional attenuation and propagation velocity are approximately constant for the \( L(0,2) \) modes, and only slightly varying for the \( L(0,1) \) and \( L(0,3) \) modes. Figures 6.18 and 6.19 show the power flux distributions across the pile radius of the \( L(0,1), L(0,2), \) and \( L(0,3) \) modes at \( \Omega = 4.5 \) and 5.3 for \( \nu = 0.18 \) and \( \nu = 0.28 \). The corresponding displacement profiles are shown in Figures 6.20 and 6.21. These mode shapes were derived based on the parameters given in Tables 6.1.

The average energy transmitted per unit time and per unit area is equal to the average power flux per unit area. Thus, one can evaluate the energy flow across the
Figure 6.18: The power flux distributions of L(0,1), L(0,2), and L(0,3) modes at $\Omega = 4.5$ and $\Omega = 5.4$ for $\nu = 0.18$ with both traction-free and embedded boundary conditions.
Figure 6.19: The power flux distributions of $L(0,1)$, $L(0,2)$, and $L(0,3)$ modes at $\Omega = 4.5$ and $\Omega = 5.4$ for $\nu = 0.28$ with both traction-free and embedded boundary conditions.
Figure 6.20: The displacement distributions of $L(0,1)$, $L(0,2)$, and $L(0,3)$ modes at $\Omega = 4.5$ and $\Omega = 5.4$ for $\nu = 0.18$ with both traction-free and embedded boundary conditions.
Figure 6.21: The power flux distributions of $L(0,1)$, $L(0,2)$, and $L(0,3)$ modes at $\Omega = 4.5$ and $\Omega = 5.4$ for $\nu = 0.28$ with both traction-free and embedded boundary conditions.
pile radius by the equivalent power flux profile. Figures 6.18 and 6.19 can be used to identify the distribution of the energy carried by the propagation wave across the pile. The distribution of the vibration amplitude of an excited wave mode across the pile radius can be evaluated by the displacement profiles given in Figures 6.20 and 6.21. For the same input power, the wave mode having a simple geometry is considered easier to excite than one with a more complicated mode shape.

The numerical mode shapes for the traction-free boundary condition are shown in part (a) and (c) of Figures 6.18, 6.19, 6.20, and 6.21. As can be seen, the power flux profile and displacement distribution across the pile radius do not exhibit significant variation between \( \Omega = 4.5 \) and 5.3. In addition to the approximately constant propagation velocity, the characteristic of invariant mode shapes further support the potential of effectively conducting frequency-controlled tests in this frequency range. Results of the numerical evaluation for waves propagating in the frequency range between \( \Omega = 4.5 \) and 5.3 for \( \nu = 0.18 \) and 0.28 indicate:

1. The energy of the propagation wave transmitted by the \( \text{L}(0,1) \) and \( \text{L}(0,2) \) mode are concentrated in the outer radius of the pile while the \( \text{L}(0,3) \) mode is concentrated in the inner radius of the pile.

2. The axial vibration amplitudes of both the \( \text{L}(0,1) \) and \( \text{L}(0,2) \) mode are more significant in the outer radius than in the inner radius.

3. The axial vibration amplitudes of the \( \text{L}(0,3) \) mode in this frequency range are essentially zero across the pile compared to the other two wave modes. The energy carried by this wave mode results in the radial vibration rather than the
axial vibration.

Numerical mode shapes of the embedded boundary condition are shown in part (b) and (d) of Figures 6.18 through 6.21. The power flux profiles of the L(0,1) and L(0,3) modes are approximately the same between Ω=4.5 and 5.3. For the L(0,2) mode, the power flux distribution near the pile perimeter does not change, but the energy within in the inner part of the piles increased at the higher frequency. Results of the numerical evaluation of the embedded piles for waves propagating in the frequency range between Ω=4.5 and 5.3 for ν=0.18 and 0.28 are summarized as below:

1. The energy of the propagation wave transmitted by the L(0,1) and L(0,2) modes is concentrated in the outer part of the pile radius while the L(0,3) mode is concentrated in the inner part of the pile radius. However, as the frequency increased from Ω=4.5 to 5.3, the L(0,2) mode of propagation wave tends to constrain more energy in the inner part of the radius.

2. The axial vibration amplitudes of the L(0,2) mode are more significant in the outer part of radius than in the inner part of radius while the axial vibration amplitudes of the L(0,3) mode are more significant in the inner part of the radius than in the outer part of the radius.

3. Although the power flux profile of the L(0,1) mode shows the tendency that the energy carried by this mode is concentrated near the pile perimeter, the axial vibration amplitude of the L(0,1) mode is essentially zero. The embedded boundary condition causes radial vibration rather than the axial vibration in this frequency range.
For the finite length prototype piles subjected to the transient frequency-controlled excitation waves, tests are conducted based on the following assumptions, in addition to the results of the numerical analysis.

1. Better quality signals can be acquired if the measurement location is near the antinode of the displacement profile where the maximum vibration amplitude occurs.

2. With the same input energy, the wave mode having simple geometry of mode shape is easier to be excited and reach the expected amplitude without the requirement of a long propagation distance.

3. For waves with more complicated mode shape, the excitation of the wave mode requires the benefit of a long propagation distance.

4. Because wave reflection is the result of an impedance change, a small-sized defect can be detected by a wave mode with the effective area of the power flux profile concentrated in the location of this defect.

Table 6.9 lists the values of Poisson's ratio at the selected frequency range previously given in Table 6.8, and the frequencies corresponding to $\Omega = 4.5$ and 5.3 for piles A06-091, A08-122, and B10-222. Ideally, the integrity of a pile can be evaluated by exciting a waveform with central frequencies within these frequency ranges. However, the frequencies that control the input waveform also are affected by the performance of the vibration shaker which is independent of the pile structure and the mounting method. For a practical application, the frequency of the input waveform needs to be selected based on both the performance of the vibration shaker and the results
of numerical analysis. A general strategy is to select the frequencies that the vibration shaker can be operated efficiently and within the frequency ranges preferable for guided wave tests.

Table 6.9: Selected Frequencies for Mode Shape Analysis

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>Poisson’s ratio, ( \nu )</th>
<th>( c_T ) (m/s)</th>
<th>( \Omega=4.5 )</th>
<th>( \Omega=5.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>0.18</td>
<td>2260</td>
<td>21241</td>
<td>25017</td>
</tr>
<tr>
<td>A08-122</td>
<td>0.28</td>
<td>2538</td>
<td>17890</td>
<td>21071</td>
</tr>
<tr>
<td>B10-222</td>
<td>0.18</td>
<td>2733</td>
<td>15412</td>
<td>18152</td>
</tr>
</tbody>
</table>

6.6 Mode Identification Method

The integrity of a pile can be evaluated by comparing the experimental results and the expected results based on numerical analysis. The wave modes which can be excited in a practical application can be forecast from the concepts presented in the previous section. To verify the reliability of the forecast technique, an experimental method was developed to identify the mode attribute of the excited waves.

Figure 6.22 illustrates the flow chart for the mode identification procedure. This identification technique is composed of two categories: preliminary work and analysis of the acquired waveform. The preliminary work starts from collecting and analyzing construction records, site and material characterization data. Based of the results of the preliminary work, the non-dimensional dispersion curves, the expected
travelling distance for the excited propagation waves, and the shear wave velocity can be determined. The approach for developing the theoretical non-dimensional dispersion curves has been presented in Sections 5.4 and 6.3. The non-dimensional group velocity dispersion curves are the basis for identifying the mode attribute of the experimentally-acquired waveform. The expected propagation distance can be found in the construction records and generally is twice of the pile length. As will be presented subsequently, the expected travel distance is essentially the conversion factor between the time coordinate and the velocity coordinate. The shear wave velocity can be determined by the methods introduced in Sections 5.6, 5.7, or 6.4. As will be introduced later, the measured waveforms can be compared directly with the numerically-derived non-dimensional group velocity dispersion curves through the normalization procedure using the predetermined shear wave velocity.

The identification of the mode attribute of the experimentally-acquired waveforms is performed by the following procedure:

1. Transform the measured waveform to the spectrogram in the joint time-frequency domain by applying the Short Time Fourier Transform (STFT) algorithm. This spectrogram is represented as spectrogram-I in Figure 6.22 where the coordinates of three axes are time, frequency, and STFT amplitude, \( \tilde{A}_{ij} \). The data array of the spectrogram-I is written as

\[ a_{ij} = a(t_i, f_j, \tilde{A}_{ij}) \]  
(6.22)

2. Convert the time coordinate, \( t_i \), of the Spectrogram-I to the velocity coordinate,
Figure 6.22: The Flow Chart of Mode Identification Procedure
\( c_i \), by the predetermined propagation distance, \( 2\ell \), by

\[
c_i = \frac{2\ell}{t_i - t_0}
\]  \hspace{1cm} (6.23)

The converted spectrogram is termed Spectrogram-II in Figure 6.22.

3. Normalize the velocity and frequency coordinates of Spectrogram-II by the predetermined shear wave velocity using

\[
C_i = \frac{c_i}{c_r}
\]  \hspace{1cm} (6.24)

\[
\Omega_j = f_j \cdot \frac{2\pi a}{c_r}
\]  \hspace{1cm} (6.25)

The normalized spectrogram is termed Spectrogram-III in Figure 6.22.

4. Compare the mode attributes of the acquired waveform with the numerically-derived non-dimensional group velocity dispersion curves with the spectrogram-III. This approach is summarized herein:

(a) Impose the numerically-derived non-dimensional group dispersion curves on the same plot with the spectrogram-III.

(b) Identify whether or not the spectrogram-III at frequencies of interest match the non-dimensional group velocity dispersion curves. The mode attribute of the acquired waveform is the same as the matched non-dimensional group velocity dispersion curve.

(c) If there are two or more non-dimensional group velocity dispersion curves that match the spectrogram-III at the frequencies of interest, compare the
trend of the velocity change with frequency between the measured waveform and the numerically-derived dispersion curves in the near frequency range.

(d) The mode attribute of the measured waveform can be determined if the velocity and dispersion trend between the spectrogram-III and the numerically-derived group velocity dispersion curves match.

6.7 Applicability of the Frequency-Controlled Method for Evaluating Pile Integrity

The proposed mode identification method is evaluated by considering results of guided wave tests on the prototype piles. Table 6.10 listed the prototype piles selected for the evaluation. These piles were designed to be intact with uniform cross sectional area or contain single defect. The details of the piles containing defected were given in Figure 5.1. The effect of geometrical attenuation on mode excitation and wave propagation is evaluated by embedded and traction-free boundary conditions.

The guided wave experiments reported herein were conducted by simultaneously acquiring the input waveforms and surface acceleration at different locations from 3 or 4 analog input (AI) channels, depending on the size of the pile radius. The following control parameters for the data acquisition system were selected based on the required time and frequency resolution for subsequent data analysis:

1. Sampling rate: 250kHz.
Table 6.10: Selected Prototype Piles for Evaluating Mode Identification Method

<table>
<thead>
<tr>
<th>Pile No.</th>
<th>Features</th>
<th>Boundary Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>Intact</td>
<td>Traction-free</td>
</tr>
<tr>
<td>A08-122</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B10-222</td>
<td></td>
<td>Embedded</td>
</tr>
<tr>
<td>B12-240</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A06-101</td>
<td>Contains a notch</td>
<td>Traction-free</td>
</tr>
<tr>
<td>A06-131</td>
<td>Contains a discontinuity</td>
<td></td>
</tr>
</tbody>
</table>

2. Number of samples to acquire at each input channel: 8192.

3. Number of Pretrigger scans: 125 data points.

4. Trigger level: 0.1 Volts.

The acquired waveform was visually examined. If it appeared to be valid, the data were accepted and stored in the buffer memory for averaging. The acquired waveforms were averaged and stored in the computer after at least 20 acceptable waveforms were acquired.

Figure 6.23 illustrates the terminology used for describing the acquired waveforms. Figure 6.23(a) shows the input force in the driving point measured by the impedance head embedded in the WR-F7 shaker. This waveform is generated by an amplified signal composed of 8 square waves with cycle frequency \( f_c \) of 15kHz. Figure 6.23(b) shows the acquired acceleration waveforms measured at the normalized radius, R,
equal to 0.75:

\[ R = \frac{r}{a} \]  

(6.26)

where \( r \) is the distance from the center of the pile, and \( a \) is the radius of the pile. The first arrival waveform is considered to be the superimposed waveforms of the wave front of the longitudinal waves, the surface wave, and the waves reflected from the edge of the pile head. The first reflection waves are the first reflections of the excited waves from the bottom of the pile. For this particular case, the first reflection waves contain two wave pockets. The mode attribute for each of the wave pockets may be different. The second wave pocket of this reflection wave has more significant signal to noise ratio (SNR) than the first wave pocket. Therefore, the second wave pocket is defined as the primary reflection wave while the first wave pocket is considered as the secondary reflection wave.

Intuitively, the propagation velocity of a wave pocket can be measured from the peak to peak intervals between the input waveforms and the reflected wave pocket. However, the frequencies corresponding to the propagation velocity measured in this manner may not be necessarily the same as the target frequency because of possible mode conversion. To evaluate the frequency content of the reflection wave pocket, the measured waveform in the time domain has to be transformed into a spectrogram in the joint time-frequency domain by a STFT algorithm. Fundamental guided wave theory indicates that the excitation of a wave mode is the result of the interaction between the propagating wave and the boundary at which the propagation wave is propagating. If a wave mode can be excited instantaneously after a vibration is generated, the travel time of the first reflection waves can be approximated by the time
Figure 6.23: An example of Input and Acquired Waveforms: (a) The input waveform measured from the impedance head of the shaker (b) The acquired waveform measured at R=0.75
intervals between the peaks of the reflection waveform and the input waveform.

The applicabilities of the numerical mode forecast and mode identification techniques are evaluated by the frequency-controlled method on intact prototype piles with traction-free boundary condition, intact prototype piles with embedded boundary condition, and defective prototype piles with traction-free boundary condition.

6.7.1 Intact Pile with Traction-free Boundary Condition

The prototype piles A06-091 and A08-122 are selected to evaluate the applicability of the proposed guided wave method. A propagation wave is excited by an input waveform composed of 8 square waves with central frequency \( f_c \)=22.5kHz. This input frequency is equivalent to the non-dimensional frequency \( \Omega \)=4.77 and 5.66 for piles A06-091 and A08-122, respectively. This frequency was selected based on the considerations of the performance of the vibration shaker as presented in Section 6.2.2 and the dispersion characteristic of the group velocities. As shown in Figure 6.9, the group velocities of the wave modes at frequencies close to the non-dimensional frequencies of 4.77 and 5.66 are approximately constant.

Required parameters for subsequent data processing, including the shear wave velocity, Poisson's ratio, pile dimensions, were derived from the results of preliminary analyses presented in Sections 5.2 and 6.4. The group velocity dispersion curves have been numerically developed in Sections 5.4 and 6.3. Table 6.11 lists these predetermined parameters.
Table 6.11: The Predetermined Parameters for Traction-free Piles A06-091 and A08-122

<table>
<thead>
<tr>
<th>Pile</th>
<th>Length, $\ell$ (m)</th>
<th>radius, $a$ (mm)</th>
<th>$c_T$ (m/s)</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A06-091</td>
<td>0.91</td>
<td>76</td>
<td>2260</td>
<td>0.18</td>
</tr>
<tr>
<td>A08-122</td>
<td>1.22</td>
<td>102</td>
<td>2538</td>
<td>0.28</td>
</tr>
</tbody>
</table>

Results of the frequency-controlled test on intact prototype piles A06-091 and A08-122 in terms of time domain waveform, joint time-frequency spectrogram, and normalized joint velocity-frequency spectrogram are shown in Figures 6.24 and 6.25. Parts (a), (b), and (c) of Figures 6.24 and 6.25 present the input waveform measured at $R=0$, and the responses of surface vibration measured at $R=0.5$ and 0.75, respectively. These waveforms were filtered by the Butterworth filter with a passband from 10kHz to 30kHz to remove unwanted signals. Parts (d) and (e) of Figures 6.24 and 6.25 show the spectrograms of the acquired waveforms in the joint time-frequency domain which were transformed by the STFT algorithm. The frequency content of the primary reflection waveforms can thus be identified from these spectrograms. The mode attribute of the primary reflection waves were identified from parts (f) and (g) of Figures 6.24 and 6.25. The conversion and normalization of the acquired waveforms were performed by the procedure suggested in Section 6.6. Tables 6.12 and 6.13 summarize the results of the frequency-controlled tests on prototype pile A06-091 and A08-122. The values of the central frequency of the responding waveforms are close to the input waveform and approximately in the range that the group velocities are constant or only vary slightly. The information in the figures suggest:
1. The mode attributes of the excited waveforms are identifiable at both \( R = 0.5 \) and 0.75 when the central frequency of the input waveform is \( \Omega = 4.77 \) (Figure 6.24). However, the mode attribute of the excited waveform is unidentifiable at \( R = 0.5 \) when the central frequency of the input waveform is close to \( \Omega = 5.66 \) (Figure 6.25).

2. The \( L(0,1) \) mode is the only mode observable within the first primary reflection waveforms at the selected frequencies.

Table 6.12: Results of Frequency-Controlled Method on Prototype Pile A06-091: Input Frequency \( f_c = 22.5 \text{kHz} \ (\Omega = 4.77) \)

<table>
<thead>
<tr>
<th>( R )</th>
<th>( f_c ) of the Primary Reflection Wave</th>
<th>Velocity ( c ) (m/s)</th>
<th>Normalized Velocity ( C )</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>23</td>
<td>1820</td>
<td>0.81</td>
<td>( L(0,1) )</td>
</tr>
<tr>
<td>0.75</td>
<td>23.2</td>
<td>1791</td>
<td>0.79</td>
<td>( L(0,1) )</td>
</tr>
</tbody>
</table>

Table 6.13: Results of Frequency-Controlled Method on Prototype Pile A08-122: Input Frequency \( f_c = 22.5 \text{kHz} \ (\Omega = 5.66) \)

<table>
<thead>
<tr>
<th>( R )</th>
<th>( f_c ) of the Primary Reflection Wave</th>
<th>Velocity ( c ) (m/s)</th>
<th>Normalized Velocity ( C )</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>22.8</td>
<td>1906</td>
<td>0.75</td>
<td>NI*</td>
</tr>
<tr>
<td>0.75</td>
<td>22.8</td>
<td>2136</td>
<td>0.84</td>
<td>( L(0,1) )</td>
</tr>
</tbody>
</table>

*Not Identifiable

The excited propagation waves were the \( L(0,1) \) mode based on the comparison between the experimental results and the numerically-derived group velocity dispersion curves. Because the numerical dispersion curves imply there are more than one mode
Figure 6.24: Results of Frequency-controlled test on pile A06-091: (a) Input waveform with $f_c=22.5\text{kHz}$, (b) Responding vibration measured at R=0.5, (c) Responding vibration measured at R=0.75, (d) The spectrogram of the responding vibration measured at R=0.5, (e) The spectrogram of the responding vibration measured at R=0.75, (f) The normalized spectrogram of the responding vibration measured at R=0.5 superimposed by the non-dimensional group velocity dispersion curves, (g) The normalized spectrogram of the responding vibration measured at R=0.75 superimposed by the non-dimensional group velocity dispersion curves.
Figure 6.25: Results of the Frequency-controlled test for pile A08-122: (a) Input waveform with $f_c=22.5$kHz, (b) Responding vibration measured at $R=0.5$, (c) Responding vibration measured at $R=0.75$, (d) The spectrogram of the responding vibration measured at $R=0.5$, (e) The spectrogram of the responding vibration measured at $R=0.75$, (f) The normalized spectrogram of the responding vibration measured at $R=0.5$ superimposed by the non-dimensional group velocity dispersion curves, (f) The normalized spectrogram of the responding vibration measured at $R=0.75$ superimposed by the non-dimensional group velocity dispersion curves.
of propagation wave existing at the frequency of interest, it is necessary to evaluate the mode that is supposed to be excited. The uncertainty of the interpretation can be prevented if the mode attribute of the excited waves can be forecast. The forecast of the numerical mode has been introduced in Section 6.5. Figure 6.26 shows the numerically-derived power flux and displacement distributions of the L(0,1), L(0,2), and L(0,3) branches at $\Omega=4.77$ and 5.66. The expected vibration response can be graphically evaluated by these mode shapes. Results of the evaluation indicate:

1. The L(0,1) and L(0,2) modes can be potentially excited by the frequency-controlled input waveform with central frequencies $\Omega=4.77$ and 5.66 based on the consideration of the complexity of geometry and the significance of vibration amplitude. The L(0,3) mode will not be excited at or near the frequencies of 4.77 and 5.66 because the computed vibration amplitude is essentially zero.

2. For the prototype piles without the benefit of long propagation distance, the amplitude of the L(0,2) mode wave may not be mobilized with the first reflection distance because of the more complicate geometry of the mode shape as shown in Figure 6.26.

Fundamental elastic wave theory indicates the mode shape does not travel with a propagating wave along a wave guide (Achenbach, 1973). Based on this consideration, the measurement location should be selected at or near an antinode of the displacement profile at which the waveforms will have the maximum vibration amplitude. For a transient wave excited by a point vibration source travelling along a finite length wave guide such as the piles illustrated herein, the surface particle remains at rest before the arrival of the excited waves, starts to vibrate as the arrival of the excited
Figure 6.26: The mode shapes of the L(0,1), L(0,2), and L(0,3) branches at $\Omega=4.77$ and 5.66
propagation wave, and damps when the propagation waves pass through the measurement location. The mode shape does not travel with the propagation waves. Hence, the expected numerical mode shape may not be fully mobilized for waves that only travel a short propagation distance. For a mobilized propagation wave, the identifiable of the mode attribute is affected by the significance of the vibration amplitude. For example, for the pile A08-122 subjected to input vibration with $f_c=22.5\text{kHz}$, the mode attribute of the first reflection wave, as illustrated in Figure 6.25(f), is more difficult to interpret because the corresponding numerical vibration amplitude at the measurement location $R=0.5$ is much smaller than at $R=0.75$.

That the $L(0,1)$ mode rather than the $L(0,2)$ mode was observed in the first reflection waveforms implies the relative difficulty in exciting a wave mode is controlled by the complexity of the geometry of mode shape, rather than the relative magnitude of the expected vibration amplitude. Experimental results suggest the $L(0,1)$ mode can be mobilized with a very short propagation distance while the $L(0,2)$ mode may require the benefit of longer propagation distance.

The integrity of the pile A06-091 and A08-122 has been verified by the results of impulse response test as presented in Section 5.6 and frequency-controlled tests presented herein. The mode attributes of the experimentally acquired waveform are consistent with the results of the numerical mode forecast. The mobilization of a wave mode in a short distance is controlled by the complexity of the geometry of the mode shape rather than the relative magnitude. The measurement should be conducted at or near the location where the numerical vibration amplitude is the most significant.
such that the mode attribute of the excited waveforms can be identified.

6.7.2 Intact Pile with Embedded Boundary Condition

The embedded prototype piles B10-222 and B12-240 were selected for evaluating the excitation of wave modes by the frequency-controlled method and the effect of geometrical attenuation for embedded boundary condition. The relatively longer Group B piles provide the round trip propagation distance such that the mode with more complex mode shape can be excited before the energy carried by the propagation wave dissipated. The propagation waves were excited by an input waveform composed of 8 square waves with a central frequency of 15kHz, equivalent to the non-dimensional frequencies of 4.38 and 5.33 for piles B10-222 and B12-240, respectively, based on the shear wave velocity determined from the UMF approach presented in Section 6.4. The group velocities of the L(0,1), L(0,2), and L(0,3) modes near these selected frequencies are approximately constant.

The shear modulus ratio, density ratio, and Poisson's ratios of pile and soil were predetermined in Sections 5.2, 5.3, and 6.4, and are listed in the below:

1. Shear Modulus Ratio, \( \frac{\rho_T}{\rho_s} = 740 \)

2. Density Ratio, \( \frac{\rho_T}{\rho_s} = 1.5 \)

3. Poisson's ratio of soil, \( \nu_s = 0.3 \)

The theoretical non-dimensional dispersion curves as presented in Sections 5.4 and 6.3 were derived from these predetermined parameters. Table 6.14 lists the dimensions,
shear wave velocities, and Poisson's ratio for the prototype piles for the subsequent waveform conversion and normalization.

Table 6.14: Predetermined Parameters for Embedded Piles B10-222 and B12-240

<table>
<thead>
<tr>
<th>Pile</th>
<th>Length (m)</th>
<th>radius, a (mm)</th>
<th>c_T (^a) (m/s)</th>
<th>(\nu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B10-222</td>
<td>2.22</td>
<td>127</td>
<td>2733</td>
<td>0.18</td>
</tr>
<tr>
<td>B12-240</td>
<td>2.40</td>
<td>152</td>
<td>2719</td>
<td>0.16</td>
</tr>
</tbody>
</table>

\(^a\) Determined from UMF approach

Figures 6.27 and 6.28 show the results of the frequency-controlled test on piles B10-222 and B12-240 in terms of the input waveform, acceleration of surface vibrations, spectrograms, and the normalized spectrograms superimposed by the numerically-derived group velocity dispersion curves. At least 20 acquired waveforms were averaged to minimize the effects of noise. The averaged waveforms were filtered by the Butterworth filter with a passband from 10kHz to 30kHz to remove unwanted signals. Part (a) of Figures 6.27 and 6.28 show the response of input waveform measured in the impedance head of the vibration shaker. Parts (b), (c), and (d) of Figures 6.27 and 6.28 show the accelerations measured on the top of the pile at R=0.25, 0.5 and 0.75. The processed time-domain waveforms were thereafter transformed to spectrograms in the joint time-frequency domain as shown in parts (e), (f), and (g) of Figures 6.27 and 6.28 such that the frequency content of the reflection waves can be identified. The spectrograms were converted and normalized to the waveforms in the
joint velocity-frequency domain as shown in parts (h), (i), and (j) of Figures 6.27 and 6.28. By comparing with the superimposed numerically-derived non-dimensional group velocity dispersion curves, the mode attribute can be distinguished.

Tables 6.15 and 6.16 summarize the results of the frequency-controlled tests on piles B10-222 and B12-240, respectively. The mode attributes of the primary reflection waves were identified by comparing the velocity and dispersion trend between the normalized spectrograms and the numerically-derived non-dimensional group velocity dispersion curves. One of the advantages of the proposed mode identification method is illustrated by the data interpretation for the waveform acquired at R=0.75 for pile B10-222. Even though the reflection wave was not clearly identifiable from Figure 6.27(d), the mode attribute of this wave can still be distinguished from the normalized spectrogram in Figure 6.27(j). The density of the reflection wave spectrogram concentrates near the L(0,1) and L(0,3) branches of dispersion curves which represent the potentially modes for the measured wave at R=0.75. Because the numerically-derived attenuation dispersion curve shows the L(0,1) mode waves at this frequency will dissipate before reflect back due to high attenuation coefficient, the mode attribute of this reflection wave should be be the L(0,3) mode.

The attenuation of the waves propagating along the prototype piles was approximated by the amplitude ratio in natural scale between the first reflection wave and the first arrival wave excited by the input vibrations. The attenuation coefficient was computed by dividing the measured attenuation by the propagation distance. To compare with the numerically-derived attenuation coefficient, the experimentally-derived
Figure 6.27: Results of the Frequency-controlled test for Pile B10-222. 
(a) Input waveform with $f_c=15$ kHz; (b), (c), (d): The responding vibration measured at $R=0.25$, 0.5, and 0.75, respectively; 
(e), (f), (g): The spectrogram of the responding vibration measured at $R=0.25$, 0.5, and 0.75, respectively; 
(h), (i), (j): The normalized spectrogram of the responding vibration measured at $R=0.25$, 0.5, and 0.75 superimposed by the non-dimensional group velocity dispersion curves.
Figure 6.28: Results of the Frequency-controlled test for Pile B12-240. (a) Input waveform with $f_c=15\text{kHz}$; (b), (c), (d): The responding vibration measured at $R=0.25$, $0.5$, and $0.75$, respectively; (e), (f), (g): The spectrogram of the responding vibration measured at $R=0.25$, $0.5$, and $0.75$, respectively; (h), (i), (j): The normalized spectrogram of the responding vibration measured at $R=0.25$, $0.5$, and $0.75$ superimposed by the non-dimensional group velocity dispersion curves.
Table 6.15: Results of the frequency-controlled method on prototype pile B10-222: propagation velocity and mode attribute

<table>
<thead>
<tr>
<th>R</th>
<th>Reflection Waves</th>
<th>Group Velocity</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_c$(kHz)</td>
<td>$\Omega$</td>
<td>c(m/s)</td>
</tr>
<tr>
<td>0.25</td>
<td>14.5</td>
<td>4.23</td>
<td>1888</td>
</tr>
<tr>
<td>0.5</td>
<td>14.5</td>
<td>4.23</td>
<td>1891</td>
</tr>
<tr>
<td>0.75</td>
<td>15</td>
<td>4.38</td>
<td>NI</td>
</tr>
</tbody>
</table>

Table 6.16: Results of the frequency-controlled method on prototype pile B12-240: propagation velocity and mode attribute

<table>
<thead>
<tr>
<th>R</th>
<th>Reflection Waves</th>
<th>Group Velocity</th>
<th>Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f_c$(kHz)</td>
<td>$\Omega$</td>
<td>c(m/s)</td>
</tr>
<tr>
<td>0.25</td>
<td>15</td>
<td>5.28</td>
<td>2569</td>
</tr>
<tr>
<td>0.5</td>
<td>15</td>
<td>5.28</td>
<td>2317</td>
</tr>
<tr>
<td>0.75</td>
<td>15</td>
<td>5.28</td>
<td>3709</td>
</tr>
</tbody>
</table>
attenuation coefficient was normalized by multiplying by the pile radius, and then reduced by the amount of the predetermined material damping coefficient $-4.14 \times 10^{-3}$, as presented in Section 5.7. Tables 6.17 and 6.18 list the results of the computed non-dimensional attenuation coefficients. For pile B10-222, the measured attenuation coefficients at $R=0.25$ and 0.5 are approximately the same which imply the mode attributes of these waves are identical.

Table 6.17: Results of the frequency-controlled method on prototype pile B10-220: Attenuation

<table>
<thead>
<tr>
<th>$R$</th>
<th>$A_1$ (Volts)</th>
<th>$A_2$ (Volts)</th>
<th>$\xi_i$ (neper/m)</th>
<th>$\xi_i a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>1.802</td>
<td>0.205</td>
<td>-0.4896</td>
<td>-0.062</td>
</tr>
<tr>
<td>0.5</td>
<td>0.928</td>
<td>0.073</td>
<td>-0.5720</td>
<td>-0.073</td>
</tr>
</tbody>
</table>

Table 6.18: Results of the frequency-controlled method on prototype pile B10-220: Attenuation

<table>
<thead>
<tr>
<th>$R$</th>
<th>$A_1$ (Volts)</th>
<th>$A_2$ (Volts)</th>
<th>$\xi_i$ (neper/m)</th>
<th>$\xi_i a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>8.913</td>
<td>0.663</td>
<td>-0.5414</td>
<td>-0.0825</td>
</tr>
<tr>
<td>0.5</td>
<td>4.907</td>
<td>0.666</td>
<td>-0.4160</td>
<td>-0.0624</td>
</tr>
<tr>
<td>0.75</td>
<td>0.569</td>
<td>0.15</td>
<td>-0.1390</td>
<td>-0.0212</td>
</tr>
</tbody>
</table>

Figure 6.29 shows the experimentally-derived group velocities and attenuation coefficients imposed by the numerically-derived group velocity and attenuation dispersion curves. For pile B10-222, both the measured group velocities and attenuation
coefficients are consistent with the L(0,3) mode dispersion curve which support mode identification results. For pile B12-240, the measured group velocities are consistent the numerical dispersion curve. However, the attenuation coefficients do not agree with the expected trend, perhaps as a results of the material damping coefficient not being constant at higher frequencies (e.g. Tharmaratnam and Tan 1990, Saniie and Bilgutay 1986).

Figure 6.30 shows the numerically-derived power flux distribution and the displacement distribution of the L(0,1), L(0,2), and L(0,3) modes of waves under embedded boundary condition at the non-dimensional frequency Ω=4.23 and 5.28 which are equivalent to the central frequencies of the reflection waves at \( f_c = 14.5 \text{kHz} \) and 15kHz for prototype piles B10-222 and B12-240, respectively. For traction-free piles, the relative difficulty in exciting a wave mode is affected by the complexity of the geometry of the mode shape and the relative magnitude of the expected vibration amplitude. The propagation or evanescence of the excited wave mode, however, is dependent on the geometrical attenuation if the pile is subject to embedded boundary condition. As shown in Figure 6.29, the L(0,1) and L(0,3) branches of theoretical group velocity dispersion curves have approximately the same propagation velocity at the non-dimensional frequencies Ω=4.23 and 5.28 which implies if the L(0,1) and L(0,3) mode waves are both excited, the propagation waves will mix together to form a wave pocket and can not be identified individually. However, the coefficient of the geometrical attenuation for the L(0,1) mode are much larger than the L(0,3) mode, which suggests the reflecting most likely the L(0,3) mode.
Figure 6.29: Comparison between the measured and numerically-derived group velocities and attenuation coefficients for piles B10-222 and B12-240
Figure 6.30: The mode shapes of the L(0,1), L(0,2), and L(0,3) mode at non-dimensional frequencies $\Omega=4.23$ and 5.28.
Results of the mode identification for pile B10-222 show the primary reflection signals at \( R = 0.25, 0.5, \) and 0.75 were identified as the \( L(0,3) \) mode, consistent with the results of numerical analysis. Results of the mode identification for pile B12-240 show the primary reflection signals at \( R = 0.25, 0.5, \) and 0.75 were identified as the \( L(0,3), L(0,1), \) and \( L(0,2) \) mode, respectively. For the signals measured at \( R = 0.5, \) the primary reflection wave is identified as the \( L(0,1) \) mode rather than the \( L(0,3) \) mode. As shown in Figure 6.30(d), the vibration amplitude of the \( L(0,3) \) mode is essentially zero at \( R = 0.5. \) With the absence of the \( L(0,3) \) mode, the \( L(0,1) \) mode will constitute the primary reflection wave because the energy carried by the \( L(0,1) \) wave mode has not been fully dissipated into the surrounding soil within the round trip propagation distance provided by the pile B12-240. For the signals measured at \( R = 0.75, \) the primary reflection wave is the second reflection of the \( L(0,2) \) mode as shown in Figure 6.28(c) and (g). The power profile in Figure 6.30(c) suggest the \( L(0,2) \) mode propagation wave mostly distributes over the outer radius of the pile while the \( L(0,3) \) mode primarily distributes over the inner radius of the pile. This results in the waveforms acquired at \( R = 0.75 \) to be the \( L(0,2) \) mode.

6.7.3 Defective Piles

Conventional non-destructive testing methods for pile foundations cannot detect small size reflection sources or multiple reflection sources that are very close to each other. The examples shown in Section 5.6.2 indicate although the impulse response method was capable of identifying the tips of both traction-free and embedded piles, the type of the defect cannot be identified from the results of impulse response method. This
is because the longitudinal propagating waves excited by a hammer-impact are essentially composed of the \( L(0,1) \) mode waves with large wavelengths and uniform displacement profiles in the lower frequency range. The wavelength of the propagation wave has to be "small" so that a defect which is very close to the primary reflection source can be distinguished. The theoretical guided wave approach shows the power flux and the displacement of a wave mode at higher frequencies are not uniformly distributed over the radius. The energy transmitted by the wave mode tends to concentrate in certain parts of the pile. When the higher frequency waves impinge upon a small defect, certain modes of the propagation wave will reflect back. By taking the advantage of this characteristic, the type of defect in a pile can be distinguished.

**Pile no. A06-101: Notch**

The prototype pile A06-101 is 1.01m long with a 76mm radius and a circumferential 25mm deep notch located 20cm above the bottom of the pile. Results of impulse response tests in Section 5.6.2 indicate the shear wave velocity of the pile is between 2306m/s and 2368m/s. The value of the shear wave velocity of this pile is computed to be 2316m/s by the UMF approach, as will be presented later. Although the location of the tip of the pile was determined from the results of the impulse response method, neither the location nor the type of the defect could be identified. To identify this particular defect, 15kHz and 25kHz waves excited by the frequency-controlled method were used to evaluate the applicability of the guided wave approach to identify the defect in this pile. These input frequencies were selected based on the consideration that the energy carried by the expected numerical wave modes is distributed over
different part of the piles as will be discussed subsequently.

Figure 6.31 shows the numerically-derived non-dimensional phase velocity dispersion curves and the imposed phase velocities computed from the resonant frequencies of the acceleration spectrum. The UMF is identified at $f=12575\text{Hz}$ such that the shear wave velocity is computed as $2316\text{m/s}$. Results of the comparison of the blue circle and black diamond with the numerically-derived dispersion curves as illustrated in Figure 6.31(a) shows these resonant peaks are primarily caused by the reflection waves from the bottom of the pile rather than the notch. However, the minor resonant effects caused by reflection of energy from the notch causes variations between the measured phase velocities and the numerically-derived phase dispersion curves.

$f_c=25\text{kHz}$ ($\Omega=5.17$)

Figure 6.32 shows the numerically-derived power and displacement profiles of the L(0,1), L(0,2), and L(0,3) modes at non-dimensional frequency $\Omega=5.17$, which is equivalent to the central frequency, $f_c=25\text{kHz}$, of the input waveform. The wavelengths of the L(0,1) and L(0,2) modes corresponding to this input frequency were numerically-computed as 0.09m and 0.23m, respectively. The power flux distribution as shown in Figure 6.32(a) suggests the energy carried by L(0,1) and L(0,2) mode propagation waves will concentrate near the perimeter of the pile such that the propagation waves will reflect back from the circumferentially discontinuous "notch" rather than the bottom of the pile. The significant vibration amplitudes of the L(0,1) and L(0,2) mode waves as shown in Figure 6.32(b) imply both the modes potentially can
Figure 6.31: Pile no. A06-101: (a) The phase velocities are computed based on the measured resonant frequencies and the total length of the pile (circle) and the length from the top to the notch of the pile (diamond). The normalization is conducted by the shear wave velocity determined from the UMF identified from the spectrum in (b).
be excited at this frequency. However, without the benefit of long propagation distance, the L(0,1) mode will be excited prior than the L(0,2) mode because its mode shape is less complicated in geometry.

Figure 6.33 shows the results of the frequency-controlled test on pile A06-101 for the input frequency \( f_c = 25\text{kHz} \). Figure 6.33(a) shows the input force measured in the impedance head of the vibration shaker. Figure 6.33(b) and (c) show the accelerations of the surface vibration measured at \( R = 0.5 \) and \( R = 0.75 \). These waveforms were the results of the average of at least 20 acquired signals. Figure 6.33(d) and (e) exhibit the spectrograms of the waveforms in the joint time-frequency domain. The frequency content and the central time of the reflection waves can be identified from these spectrograms. Figure 6.33(f) and (g) show the normalized spectrograms in the joint velocity-frequency domain and the superimposed group velocity dispersion curves. The normalization was computed based on the assumption that the incident waves were reflection from the location of the notch.

The mode attributes of the first reflection waves A and B are both identified as the L(0,1) mode as shown in Figure 6.33(f) and (g). The central frequencies of the reflection waves A and B, however, were converted to 23.3kHz and 23.5kHz rather than the input frequency 25kHz. The group velocity of the L(0,1) mode propagation wave measured at \( R = 0.5 \) is computed as 1857m/s. The arrival time of the waveform B is not well defined in Figure 6.33(c). Nevertheless, it can be identified from the corresponding spectrogram as the density peak of waveform B in Figure 6.33(e). The group velocity of the L(0,1) mode propagation wave measured at \( R = 0.75 \) was thus
Figure 6.32: Pile no. A06-101: Mode Shapes at $\Omega=5.17$ (f=25kHz) (a) Power Distribution, (b) Displacement Distribution
Figure 6.33: Pile no. A06-101: Results of the frequency-controlled method: (a) Input waveform with $f_c=25$ kHz, (b) Responding vibration measured at $R=0.5$, (c) Responding vibration measured at $R=0.75$, (d) The spectrogram of the responding vibration measured at $R=0.5$, (e) The spectrogram of the responding vibration measured at $R=0.75$, (f) The normalized spectrogram of the responding vibration measured at $R=0.5$ superimposed by the non-dimensional group velocity dispersion curves, (f) The normalized spectrogram of the responding vibration measured at $R=0.75$ superimposed by the non-dimensional group velocity dispersion curves.
be computed as 1905 m/s.

Results of Numerical analysis suggest the L(0,2) mode can also be excited and propagates along the outer radius of the pile. However, direct evidence does not show that the L(0,2) mode has been excited. The possible causes are listed as below:

1. The first reflection of the L(0,2) mode waves is not fully mobilized and is masked by the surface responding waves of the input vibration.

2. Numerically-derived group velocity dispersion curves show the group velocity of the L(0,2) mode is approximately twice of the L(0,1) mode at $\Omega=5.17$ which implies the arrival the second reflection L(0,2) modes waves will be masked by the first reflection of the L(0,1) mode waves.

Figure 6.34 shows the spectrograms of the reflection waves normalized by the propagation distance of two round trips between pile top and the location of the notch. The comparison between the experimental results and numerical results implies the L(0,2) mode waves are masked by the waveforms A and B. The waveform C is the second reflection of the L(0,1) mode wave. Results showing that two significant L(0,1) mode reflection waves were observed at R=0.75 while only one significant L(0,1) mode reflection wave was observed at R=0.5 support the numerical forecast that the energy carried by this mode wave primarily concentrates in the outer radius of the pile. Table 6.19 summarizes the experimental results. In summary, experimental results show that the L(0,1) and L(0,2) mode waves can be excited by an input wave composed of 8 square waves and centered at $f_c=25$ kHz. The location of the notch for the pile A06-101 can be detected by the L(0,1) mode waves.
Table 6.19: The Group Velocity and Mode Attribute of the Reflection Waves Generated by 8-Square Wave with $f_c=25$kHz for A06-101

<table>
<thead>
<tr>
<th></th>
<th>$R=0.25$</th>
<th></th>
<th>$R=0.75$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$f$(kHz)</td>
<td>$c_g$(m/s)</td>
<td>$C_g$</td>
<td>Mode</td>
</tr>
<tr>
<td>1st Reflection</td>
<td>23.3</td>
<td>1857</td>
<td>0.8</td>
<td>L(0,1)</td>
</tr>
<tr>
<td>2nd Reflection</td>
<td>23.3</td>
<td>3714</td>
<td>1.6</td>
<td>L(0,2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>NI</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$f_c=15$kHz ($\Omega=3.1$)

Figure 6.32 shows that the energy associated with the L(0,1) and L(0,2) wave modes at $\Omega=5.17$ is primarily concentrated near the perimeter of the pile, and thus would be expected to be reflected from the notch. However, to sense the bottom of the pile, one would want to input energy that was concentrated in the center of the pile. To this end, a L(0,1) mode wave was excited at $\Omega=3.1$ which is equivalent to $f=15$kHz for pile A06-101. Figure 6.35 shows the power flux and displacement distribution of this mode. The wavelength of this mode wave was numerically-computed to be 0.19m. The pattern of the power flux and displacement profile implies the energy carried by this mode of propagation waves will primarily concentrates in the inner part of the radius. If the cross section inside the defect is intact, this mode propagation wave will pass through the notch and reflect off the bottom of the pile.

Figure 6.36 shows the results of the 15kHz frequency-controlled test. The wave-
Figure 6.34: Pile no. A06-101: the normalized spectrograms of the 2nd reflection waves computed based on the assumption that the reflection source is the notch and the superimposed numerically-derived group velocity dispersion curves
Figure 6.35: Pile no. A06-101: The power flux and displacement distribution of the L(0,1) mode at $\Omega=3.1$ (f=15kHz) and $\nu=0.28$ (a) Power Distribution, (b) Displacement Distribution
form of the input force and the accelerations of surface vibration were presented in Figures 6.36(a), (b), and (c). The acceleration waveforms measured at R=0.5 and 0.75 were transformed to spectrograms in the joint time-frequency domain as shown in Figure 6.36(d) and (e). The central frequencies of the reflection waves were identified to be 15kHz and 14.2kHz at R=0.5 and 0.75, respectively. The travel time of the excited wave packets can be determined from the time intervals between the peak of the reflection spectrograms and the peak of the input force. Figure 6.36(b) and (c) show the computed group velocities based on the assumption that the waves were reflection from the bottom of the pile.

To identify the mode attribute of the excited waveforms, the measured signals have to be normalized and converted into the spectrograms in the joint velocity-frequency domain so that the numerically-derived group velocity dispersion curves and the measured waveforms can be compared directly. The shear wave velocity, 2316m/s, was determined from the UMF approach. Figure 6.36(f) and (g) show the normalized spectrograms of the reflection waves acquired at R=0.5 and 0.75 and superimposed numerically-derived group velocity dispersion curves. The spectrograms were normalized based on the assumption that the waves were reflected from the bottom of the pile. Results of the comparison show the mode attributes of the excited waveforms are the L(0,1) mode waves reflected from the bottom of the pile. The waves excited by the 15kHz frequency-controlled wave with wavelength of 0.19m pass through the inner part of the defect and reflected from the bottom of the pile.

Table 6.20 summarizes the selected input frequencies for identifying the notch type
Figure 6.36: Pile no. A06-101: Results of the frequency-controlled method. (a) Input waveform with \( f_c = 15 \text{kHz} \), (b), (c) Responding vibrations measured at \( R = 0.5 \) and 0.75, respectively, (d), (e) The spectrogram of the responding vibrations measured at \( R = 0.5 \) and 0.75, respectively, (f) The normalized spectrogram of the responding vibration measured at \( R = 0.5 \) and 0.75, respectively, superimposed by the non-dimensional group velocity dispersion curves.
defect, the actually excited wave mode and the corresponding numerically-derived wave length, the inner diameter of the notch, and identified reflection source. The "notch" type defect can be identified by exciting two different modes of propagation waves, one with energy concentrated near the perimeter of the pile and another with energy concentrated near the center of the pile. For the prototype pile A06-101, the L(0,1) mode wave at $\Omega = 5.17$ with wavelength of 0.09m propagates along the outer radius of the pile. The notch, 25mm wide creates an intact pile with an inner radius of 0.1m at $\ell = 0.81$m from the pile top, and prevents the wave from passing through the defect and causes reflections from the circumferential discontinuity. The L(0,1) mode wave at $\Omega = 3.1$ with wavelength of 0.19m will pass through the inner part of the "notch" without being reflected until it encounters the tip of the pile because the wavelength of this mode wave is somewhat larger than the size of the defect and primarily propagates along the inner part of the pile.

Table 6.20: Pile A06-101: The input waves and identified reflection source

<table>
<thead>
<tr>
<th>Input frequency f(kHz)</th>
<th>Excited Mode</th>
<th>Wave Number $\xi_r,a$</th>
<th>Wavelength $\frac{2\pi}{\xi_r}$ (m)</th>
<th>Defect(^a) size (m)</th>
<th>Notch to(^b) tip (m)</th>
<th>Reflection source</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>L(0,1)</td>
<td>2.54</td>
<td>0.19</td>
<td>0.10</td>
<td>0.2</td>
<td>tip</td>
</tr>
<tr>
<td>25</td>
<td>L(0,1)</td>
<td>5.42</td>
<td>0.09</td>
<td></td>
<td></td>
<td>defect</td>
</tr>
</tbody>
</table>

\(^a\) The inner diameter of the notch
\(^b\) The distance between the notch and the pile tip
Pile No. A06-131: Discontinuity

The 1.31m long, 76mm radius prototype pile A06-131 is a complex pile composed of two separate cylinders laminated together by epoxy. The two separate parts are laminated at 31cm above the bottom of the pile. Experiment results in Section 5.6.2 show the location of the pile tip can be detected by the impulse response methods. However, the location of the discontinuity was not able to be detected because either the distance from the defect to the pile tip is too small or the defect itself is too small. The characteristics of the higher frequency wave modes, specifically the small wavelength and various mode shapes, were used to identify reflection from defect. The propagation waves were excited by selected waveforms using the frequency-controlled method.

Figure 6.37 shows the measured phase velocities at the resonant peaks of the spectrum and the superimposed numerically-derived non-dimensional group velocity dispersion curves. The measured phase velocities were normalized by the shear wave velocity determined from the UMF approach. Because the UMF is identified at $f=13362\, \text{Hz}$, the shear wave velocity $c_T$ of the pile is computed as $2461\, \text{m/s}$. Results of the comparison of the circles (assumed reflection from bottom) and diamonds (assumed reflection from discontinuity) with the numerically-derived dispersion curves as illustrated in Figure 6.31(a) shows these resonant peaks are primarily caused by the reflection waves from the bottom of the pile rather than the notch. The Poisson's ratio of this pile, however, cannot be identified from the trend of the measured phase velocities at resonant frequencies. This is caused by the minor resonant effects superimposed on the primary resonant peaks as a result of the reflection wave from the laminated section such that the measured phase velocities exhibit fluctuated up
and down the numerically-derived phase dispersion curves.

\[ f_c = 25 \text{kHz} \quad (\Omega = 4.86) \]

Figure 6.38 shows the numerically-derived power flux and displacement profiles of the L(0,1), L(0,2), and L(0,3) mode at \( \Omega = 4.86 \), equivalent to 25kHz, for the pile A06-131 based on the shear wave velocity 2461m/s, determined from the UMF approach. The wavelengths of the L(0,1) mode at this frequency is 0.09m. Based on the considerations of the geometry and amplitude of these modes, it is suggested that L(0,1) and L(0,2) mode of propagation waves potentially can be excited by the input waveform centered at \( \Omega = 4.86 \). Figure 6.38 also implies the energy carried by the L(0,1) and L(0,2) modes are both concentrated in the outer part of the radius. However, the L(0,2) mode wave is not expected to be observed in the first reflection wave because it is less likely to be fully mobilized within the propagation distance provided by pile A06-131 and would be masked by the surface responding waves of the input vibration.

Figure 6.39 shows the results of the 25kHz frequency-controlled tests including the input force, the accelerations of surface vibration measured at R=0.5 and 0.75, the spectrograms of the measured accelerations transformed by the STFT algorithm, and the normalized spectrograms in the joint velocity-frequency domain superimposed upon the numerically-derived group velocity dispersion curves. The first and second primary reflection waveforms at R=0.5 are marked as A and B in Figure 6.39(b), and the first and second primary reflection waveforms at R=0.75 are marked as C and D in Figure 6.39(c). The center frequencies of these primary reflection waveforms are
Figure 6.37: Pile no. A06-131: (a) The phase velocities are computed based on the measured resonant frequencies and the total length of the pile (circle) and the length from the top to the notch of the pile (diamond). The normalization is conducted by the shear wave velocity (2461 m/s) determined from the UMF (13362 Hz) identified from the spectrum in (b).
Figure 6.38: Pile no. A06-101: Mode Shapes at $\Omega = 4.86$ (f=25kHz) and $\nu = 0.14$
identified from Figure 6.39(d) and (e) as 25.2kHz, 24kHz, 23kHz, and 24.8kHz for waveforms A, B, C, and D, respectively.

Figure 6.39(f) and (g) show the normalized spectrogram in the joint velocity-frequency domain imposed by the numerically-derived non-dimensional group velocity dispersion curves. These normalized spectrograms, representing the first primary reflection waves, were converted from the spectrograms in Figure 6.39(d) and (e) based on the assumption that the reflection source is the discontinuity at 1.0m from the top of the pile such that the propagation distance for the first reflection wave is 2.02m. Figure 6.40(a) and (b) show the normalized spectrograms of the second reflection waves at R=0.5 and 0.75 imposed by the numerically-derived non-dimensional group velocity dispersion curves. The normalization was performed based on the assumption that the propagation distance is 4 times of the distance between the pile top and the discontinuity, 4.04m. Results of the comparison implies the mode attributes of the first reflection wave C, and the second reflection wave D measured at R=0.75 are both the L(0,1) modes which is consistent with the expected results predicted from the characteristics of the numerically-derived mode shapes.

Although the first and second reflection waves A and B measured at R=0.5 are very close to the L(0,1) branch, the mode attributes of these reflection waves are not clearly defined. The numerically-derived displacement profile on Figure 6.38 shows the expected L(0,1) mode vibration amplitude at R=0.5 is smaller than at R=0.75. Therefore, the waveforms measured at R=0.5 will have lower signal to noise ratio and easier to be interfered by ambient signals which causes errors in the measured group
Figure 6.39: Pile no. A06-131: Results of the frequency-controlled method: (a) Input waveform with $f_c=25\text{kHz}$, (b), (c) Responding vibrations measured at $R=0.5$ and 0.75, respectively, (d), (e) The spectrogram of the responding vibrations measured at $R=0.5$ and 0.75, respectively, (f) The normalized spectrogram of the responding vibration measured at $R=0.5$ and $R=0.75$, respectively, superimposed by the non-dimensional group velocity dispersion curves.
Figure 6.40: Pile A06-131: Mode attributes of the 2nd reflections of the waves excited by 25kHz frequency-controlled input wave. The reflection source is identified at the location of the discontinuity (L=1.01m).
velocities. Based on these reasons, the waveforms A and B are the L(0,1) mode. Table 6.21 summarizes the mode attributes and group velocities of the propagation waves excited by the 25kHz frequency-controlled method.

Table 6.21: The group velocity and mode Attribute of the reflection waves generated 25kHz frequency-controlled waves for pile A06-131

<table>
<thead>
<tr>
<th></th>
<th>( R=0.50 )</th>
<th>( R=0.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( f(\text{kHz}) )</td>
<td>( c_g(\text{m/s}) )</td>
</tr>
<tr>
<td>1st Reflection</td>
<td>25.5</td>
<td>2061</td>
</tr>
<tr>
<td>2nd Reflection</td>
<td>24</td>
<td>2109</td>
</tr>
</tbody>
</table>

\( f_c=15\text{kHz} \) (\( \Omega=2.91 \))

The location of the discontinuity in the prototype pile A06-131 can be detected by the propagating wave excited by the \( \Omega=4.17(f_c=25\text{kHz}) \) frequency-controlled waveform with a wavelength 0.09m. Results of the mode identification at this frequency implied no difference in geometry of the defect across the section which is consistent with the type of the discontinuity. To provide further evidence of this interpretation, waves excited by \( f_c=15\text{kHz} \), equivalent to \( \Omega=2.91 \), are selected to evaluate the integrity of the pile because: (a) the numerically-derived wavelength of the L(0,1) mode at this frequency is 0.21m, somewhat larger than the diameter of the defect, and (b) wave energy carried by this mode wave is concentrated in the inner part of the pile. The L(0,1) mode shapes with respect to \( \Omega=2.91 \) is not presented herein because it is very similar to that of \( \Omega=3.1 \), as shown in Figure 6.35.
Figure 6.41 shows the results of the frequency-controlled method for $f_c=15$kHz. Figure 6.41(b) and (c) show the primary reflection waves marks as A and B at R=0.5 and 0.75, respectively. The group velocities of A and B are computed to be 1471m/s and 1092m/s, respectively, based on the assumption that the primary reflections occur at the laminated part of the pile. These time-domain waveforms are subsequently transformed to spectrograms in the joint time-frequency domain as shown in Figure 6.41(d) and (f). The central frequencies of the waveform A and B can thus be identified as 14.2kHz and 15kHz, respectively. To identify the mode attribute of these waves, the spectrograms are normalized to the joint velocity-frequency domain so that the experimental results can be compared with the numerical results directly. As shown in Figure 6.41(f) and (g), the mode attributes of the waveforms A and B are both the $L(0,1)$ mode which reflects the fact that there is no difference in geometry across the location of the defect.

Table 6.22 summarizes the selected input frequencies for identifying the discontinuity type defect, the actually excited wave mode and the corresponding numerically-derived wave length, the size of the defect, and the identified reflection source. Reflections from the "discontinuity" were identified by exciting two modes waves, one with energy concentrated near the perimeter of the pile while another with energy concentrates near the center of the pile. Results of the experiment show that the $L(0,1)$ mode wave at $\Omega=4.86$ with wavelength of 0.09m, expected to propagates primarily along the outer radius of the pile, was reflected from the defect, as was the $L(0,1)$ mode wave at $\Omega=2.91$ with wavelength of 0.21m which propagated primarily
Figure 6.41: Pile no. A06-131: Results of the frequency-controlled method.: (a) Input waveform with $f_c=15\text{kHz}$, (b), (c) Resonating vibrations measured at $R=0.5$ and 0.75, respectively, (d), (e) The spectrogram of the resonating vibrations measured at $R=0.5$ and 0.75, respectively, (e) The normalized spectrogram of the resonating vibration measured at $R=0.5$ and $R=0.75$, respectively, superimposed by the non-dimensional group velocity dispersion curves.
along the inner part of the pile. Thus, the type of the defect can be inferred as the uniformly discontinuous cross section which is consistent with the discontinuity in the pile A06-131.

Table 6.22: Pile A06-131: The input waves and identified reflection source

<table>
<thead>
<tr>
<th>Input frequency</th>
<th>Excited Mode</th>
<th>Wave Number $\xi_a$</th>
<th>Wavelength $\frac{2\pi}{\xi_r}$ (m)</th>
<th>Defect size (m)</th>
<th>defect to tip (m)</th>
<th>reflection source</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 f(kHz)</td>
<td>2.91 L(0,1)</td>
<td>2.25</td>
<td>0.09</td>
<td>0.15</td>
<td>0.31</td>
<td>defect</td>
</tr>
<tr>
<td>25 f(kHz)</td>
<td>4.86 L(0,1)</td>
<td>5.3</td>
<td>0.21</td>
<td></td>
<td></td>
<td>defect</td>
</tr>
</tbody>
</table>

*a The diameter of the discontinuity
*b The distance between the discontinuity and the pile tip

6.8 Summary: Frequency-Controlled Method to Evaluate the Integrity of a concrete Pile

This section summarizes the procedures to evaluate pile integrity using the frequency-controlled method.

6.8.1 Apparatus

1. Vibration Shaker (see Section 4.2.2).

2. Transducer (see Section 4.2.3).

3. PC-controlled DAQ System (see Section 4.4.3).
6.8.2 Test Procedure

1. Schematic of the test set-up is shown in Figure 6.42.

2. Assemble the apparatus:

   (a) Mount the modal shaker and transducers on the top surface of the pile (see Sections 4.2.2 and 4.2.3).

   (b) Assemble the Modal Shaker, Transducer, and PC-Control System (see Section 4.5).

3. Set parameters for the frequency-controlled input waveform including the number of cycles, \( n \), central frequency, \( f_c \), point update rate, \( f_p \), and power level.
The minimum number of cycles should be 8 for better frequency control. However, the duration of the input waveform, \( \frac{n}{f_c} \), must be less than the round-trip travel time \( \Delta t \) for the excited waves.

4. Set the data acquisition parameters, including sampling rate, \( f_s \), number of channels, trigger type and level, pretrigger scans, and number of repetitions.

5. Perform the frequency-controlled test. Examine the acquired waveform. If it appears to be valid, store the data for later analysis.

### 6.8.3 Data Analysis

1. Transform the acquired waveforms into the frequency domain by FFT. Examine the frequency content of the waveform from the spectrum. Carefully remove the unwanted noises using digital filtering techniques.

2. Compute the phase velocities of the excited propagation wave from the resonance spectrum using Equation 6.5. The shear wave velocity can thereafter be determined by the UMF approach, as presented in Section 6.4.2, or by the best fit approach based on results of impulse response methods, as presented in Sections 5.6 to 5.7.

3. The travel time \( \Delta t \) of the excited waveform is determined by the difference between the peak of the reflection wave pocket, \( t_1 \), and the peak of the input waveform, \( t_0 \). The group velocity can be computed by

\[
c_g = \frac{2L}{\Delta t}
\]  

(6.27)
where $2L$ is the predetermined propagation distance. Figure 6.43 presents an example to illustrate the measurement of the travel time between the reflection wave and the input waveform where the reference peaks are the seventh peak in each waveform.

4. The integrity of the pile is evaluated by comparing the experimentally-measured group velocity with the numerically-derived group velocity. Figure 6.44 shows the flow chart for integrity evaluation using the frequency-controlled method. The time-domain waveforms acquired from multiple channels are transformed into the joint time-frequency waveforms by the Short Time Fourier Transform (STFT), and subsequently normalized into the joint velocity-frequency domain by the predetermined propagation distance, shear wave velocity, and the reference peak of the input waveform. Numerically develop the non-dimensional group velocity dispersion curves and mode shapes based on the construction record and results of site characterization. The integrity of the pile is evaluated by comparing the consistency between the experimental and numerical results. If the measured results match the numerically-derived results, the evaluation is done. Otherwise, repeat the normalization and comparison with a different value of assumed propagation distance. The location of the reflection source can be found by the trial and error. The type of defect can be identified by evaluating the numerical mode shapes, as discussed in Sections 6.7.3.
Figure 6.43: Example of travelling time computation. The frequency-controlled test was conducted on pile A08-122. The input waveform is composed of 8 square wave with central frequency $f_c = 15$ kHz and the acceleration response was measured at $R = 0.75$. Both waveforms were filtered by the Butterworth filter with passband from $5$ kHz to $30$ kHz.
Figure 6.44: The flow chart of integrity evaluation using the frequency-controlled method
Chapter 7

Summary and Conclusion

The application of the conventional impulse response and sonic echo methods in the integrity evaluation for drilled shafts are limited by the maximum useful frequency that can be generated by a hammer impact and the one-dimensional wave propagation theory-based interpretation method. To extend the frequency range for the surface reflection techniques, a three-dimensional theoretical approach, based on equations of elasticity, was developed by treating the drilled shaft as a cylindrical waveguide embedded in soil (Hanifah, 1999). The solutions of the general frequency equation for drilled shafts are expressed as the dispersion relationship between the non-dimensional frequency, $\Omega$, and non-dimensional wave number, $\xi a$. The general frequency equation accounts for all motions including flexural, longitudinal, and torsional waves in a pile (Finno et al., 2002).

The frequency equation for longitudinal modes were derived by simplifying and decomposing the general frequency equation by setting the order of special functions to zero. The solutions of the longitudinal mode frequency equation have been solved and presented by (Finno et al., 2001). Infinite numbers of solutions for the frequency
equation forms branches of dispersion relations. These transcendental dispersion relations are further developed into group velocity, phase velocity, and attenuation dispersion curves. Each solution of the longitudinal mode frequency equation is termed a "mode". The mode attribute can be characterized by the group velocity, phase velocity, attenuation coefficient and special mode shapes in terms of distributions of displacement, stress, and power flux across the waveguide radius. The power flux distribution reveals the concentration of energy transmitted by a propagating wave mode across the waveguide. Whereas wave propagation is a phenomena of energy transmission, the mode shape excited by the transmitted energy does not travel. Thus, the measurement points having optimal signal to noise level can be found from the numerically-derived mode shapes. Fundamental wave propagation theory indicates that wave reflection is a result of impedance change. If the mode attribute of a reflection can be identified, the location and type of the source affecting the impedance can thereafter be found.

Results of the theoretical evaluation indicate the following tasks need to be fulfilled before the experimental approach can be designed for integrity evaluation using the guided wave method.

- Determine the dynamic elastic properties of in-situ soils and concrete.
- Select frequency ranges and modes for evaluating the integrity of a pile.
- Generate a frequency-controlled waveform.
- Measure group and phase velocities and geometrical attenuation.
- Convert between the dimensional and non-dimensional solutions of the fre-
frequency equation.

- Identify the mode attribute of the reflected waves.

The properties of in-situ soils and concretes, which include the density ratio, shear modulus ratio, and Poisson's ratio of pile and soil, are required for deriving the dispersion relations for the longitudinal modes in a pile. Generally, the frequency range that is preferable for guided wave tests has to satisfy the following properties.

- All the wave modes along the same longitudinal branch in a selected frequency range have approximately constant group velocity and attenuation coefficient.

- The attenuation coefficients of all the wave modes along the same longitudinal branch in a selected frequency range are small in value.

- The geometry of mode shapes of all the modes along the same branch in a selected frequency range are uncomplicated and similar to each other.

- If more than one branch of dispersion curves exist in the frequency range of interest, mode characteristics between different branches need to be significantly different.

When the wave modes within a frequency range along a longitudinal branch satisfy the above criteria, the vibration response measured at the top of a pile can be interpreted. For example, the L(0,1) mode of waves excited by a hammer impact with non-dimensional frequency \( \Omega < 2 \) have constant group velocity, uniform mode shapes, and constant and low attenuation coefficient which provide the frequency range of the conventional impulse response and sonic echo methods. However, these non-dispersive characteristic (mode properties invariant with frequency) only exists
in a limited number of frequency ranges. For example, the frequency range $4 < \Omega < 6$ contains the peak group velocity of the $L(0,2)$ mode and the frequency range that the group velocity of the $L(0,1)$ mode waves are asymptotic to Rayleigh wave velocity ($\Omega > 5$).

A method to control the frequency content of the excited waveform is proposed and numerically evaluated. The input waveform is composed of a finite number of sine waves or square waves. The central frequency of the input waveform is controlled by the cycle frequency of the sine wave or square wave composed of the input waveform, while the bandwidth of the input waveform is controlled by the number of the sine or square waves. Although a narrow band of frequency control can be achieved by composing the input waveform with a large number of cycles of waves, the length of the input waveform is limited by the dispersion behavior of the excited propagation waves. A long wave packet may contain more than one mode of reflected waves that make the measurement of the propagation velocity more complicated. Furthermore, the reflected wave should not be masked by the input waveforms, especially for tests conducted on small-scale prototype piles.

A test system is built based on the requirements defined by results of numerical evaluation. This system is portable, PC-based, and is able to perform the following functions by the programmed virtual waveform generator and virtual oscilloscope.

- Frequency-controlled waveform generation.
- Multiple channel triggered data acquisition.
• Signal processing and data analysis.

The frequency-controlled waveform is generated from a vibration shaker mounted vertically in the center of the pile surface. The vibration shaker is controlled by the amplified signals edited by the virtual waveform generator. The input waveform can be measured from a force gage or an accelerometer embedded in the impedance head of the vibration shaker. Two or three accelerometers are mounted at locations across the pile on the same surface as the vibration shaker. The input waveform and surface responding vibrations in turn are measured simultaneously by the multiple channel virtual oscilloscope. The data acquisition is triggered by the vibration of input waveform.

The responding vibrations are measured in the time domain and transformed into frequency domain by the Fast Fourier Transform (FFT) algorithm and the Short Time Frequency Transform (STFT) algorithm. The group velocity of the excited waveform is measured from the time domain data while the phase velocity is computed from the resonant frequencies in the frequency domain. The spectrogram derived from the STFT algorithm can be used to identified the evolution of the frequency contents of the excited waveforms on the time basis. Various digital filtering techniques are applied to eliminate the unwanted signals outside the frequency range of interests.

In order to develop an experimental approach to evaluate the integrity of pile foundations, a series of small scale prototype piles with diameters from 152mm (6 inch) to 457mm (18 inch) and length to diameter ratio from 4 to 8.7 were constructed
in laboratory and later embedded in the National Geotechnical Experimental Site (NGES). These piles are either flawless or contain a designed defect. Numerical solutions of dispersion relations for the longitudinal modes for these piles are derived from the soil and concrete parameters determined from laboratory and in-situ tests. Because of the inherent uncertainty in finding the Poisson's ratio for concrete, dispersion curves corresponding to Poisson's ratio from 0.14 to 0.28, which represent a reasonable range for concrete, were developed.

Conventional impulse response tests were conducted on these prototype piles and the results are interpreted by the three-dimensional guided wave approach. The evidence that the guided waves are excited in the pile by a hammer impact is verified by the comparison between the experimental results and numerical results for piles with known integrity. The comparison is conducted by converting the experimental results and the numerical results to the same basis — dimensional or non-dimensional, by multiplying or dividing by the bulk shear wave velocity. The bulk shear wave velocity can be estimated by one of the following approaches depending on the maximal useful frequency.

- Assume that the measured propagation velocity equals the bar wave velocity.

- Normalize the measured phase velocities by a range of assumed shear wave velocity. Find the the shear wave velocity that gives the best fit to the numerical results.

- Identify the frequency corresponding to the universal mode for the L(0,1) branch.
Comparison of the results indicate that the hammer-generated waves are primarily the L(0,1) mode. The phase velocities measured at resonant frequencies follow the trend of the numerically-derived L(0,1) branch phase velocity dispersion curve. The measured attenuation is approximately equal to the numerically-derived geometrical attenuation. As predicted by the numerically-computed mode shapes, the amplitudes of the vibrations measured at various location across the pile exhibit the same order of magnitude.

The limitation of the impulse response method is demonstrated by the prototype piles containing a small defect or a defect close to the tip. Although the existence of an anomaly can be distinguished by comparing the results with intact piles, the location and type of these defects are not identifiable. This is an inherent disadvantage for impulse response method because the wavelength of a hammer-generated stress wave is much larger than the size of these defects. To extend the applicability of the surface reflection techniques, it is necessary to use wave modes of higher frequency.

At higher frequencies, more than one longitudinal wave mode exists at a given frequency; This makes the data interpretation more complicated. To eliminate the uncertainty, a mode identification technique is developed. The waveforms measured in the time domain are transformed to a spectrogram in the joint time-frequency domain by a STFT algorithm. The joint time-frequency spectrogram is subsequently converted and normalized to a non-dimensional joint velocity-frequency spectrogram. By comparing this spectrogram with the numerically-derived group velocity dispersion curves, the mode attribute of the reflected waves can be identified.
For waves excited by an axisymmetrical point source, such as a vibration shaker mounted at the center of the pile surface, excitation of a wave mode is a function of the complexity of the geometry of the mode shape and the relative magnitude of the vibration amplitude. Based on these considerations, the wave modes that potentially can be excited at frequencies of interest can be predicted by the numerically-derived mode shapes.

The integrity of a drilled shaft can be evaluated by comparing the wave modes actually being excited to those expected to be excited. The modes expected to be excited are predicted numerically. Because information such as the geometry and concrete strength of a drilled shaft can be collected from the design drawings or construction record, the propagation velocity of the actually excited wave mode can be computed based on the as-built length of the drilled shaft. The mode forecast approach and mode identification technique are applied on both traction-free and embedded intact prototype piles.

The location of a defect that is not identifiable by impulse response method can be detected by waves of small wavelength excited by frequency-controlled method. The type of these defects can be identified by introducing two or more wave modes into the pile and finding the reflection source for each of the excited wave modes. Because the energy flows of certain modes are concentrated near the perimeter of the pile while other modes are concentrated near the center of the pile, the reflection of these waves represents the impedance change in certain part of the reflection source.
rather the the overall cross section. Practical examples demonstrated by prototype piles with a neck and discontinuity type of defects close to the tip of the pile verify the applicability of this approach.

The applicability of the proposed frequency-controlled method on full-scale piles is limited by the level of the input power of the vibration shaker. Possible solutions include (a) developing a new vibration shaker that is able to generate high power vibrations at high frequencies, and (b) designing a multiple vibration shaker system. The former method is essentially an extensive application of the current experimental model. The later method, however, involves more complicated interface integration between system components. Nevertheless, special functions such as generating modes with controlled attributes have been proved feasible in the applications of other research fields (e.g. Shin and Rose 1998, and Zemanek 1971).
References


Davis, A. (1994). Impedance log for the length and shape measurements of drilled shafts from mobility (tdr) tests (computer program).


